Chapter 17: Decoupling of Equations of Motion

Introduction

In the study of structural dynamics and earthquake engineering, complex systems—such as multi-degree-of-freedom (MDOF) structures—require an analytical framework to evaluate their dynamic behavior under seismic excitations. These systems yield coupled differential equations of motion that are often challenging to solve directly. The method of *decoupling* simplifies the analysis by transforming the set of coupled equations into a set of independent equations. This chapter focuses on the decoupling techniques using modal analysis, properties of orthogonality, and diagonalization of mass and stiffness matrices. Understanding these principles allows for efficient and accurate seismic analysis and design of multi-storey structures.

17.1 Equations of Motion for MDOF Systems

For a linear elastic system with n degrees of freedom, the general form of the equations of motion under external excitation (e.g., earthquake-induced base motion) is:

$$[M]\{\ddot{u}(t)\} + [C]\{\dot{u}(t)\} + [K]\{u(t)\} = \{F(t)\}$$

Where:

- $[M]$ is the mass matrix
- $[C]$ is the damping matrix
- $[K]$ is the stiffness matrix
- $\{u(t)\}$ is the displacement vector
- $\{F(t)\}$ is the force vector (including earthquake forces)

This is a system of n second-order coupled differential equations.

17.2 Need for Decoupling

Solving the above system directly is computationally expensive and often impractical, especially for large structures. Decoupling transforms the coupled equations into n independent scalar equations, each corresponding to a *mode of vibration*. This enables modal superposition techniques for dynamic analysis.

17.3 Modal Transformation

Let the modal matrix $[\Phi]$ consist of n linearly independent eigenvectors:

$$\{u(t)\} = [\Phi]\{q(t)\}$$

Where:

- $[\Phi]$ is the modal matrix containing eigenvectors as columns
- $\{q(t)\}$ is the modal coordinate vector

Substituting into the equations of motion:

$$[M][\Phi]\{\ddot{q}(t)\} + [C][\Phi]\{\dot{q}(t)\} + [K][\Phi]\{q(t)\} = \{F(t)\}$$

Multiplying both sides by $[\Phi]^T$:

$$[\Phi]^T[M][\Phi]\{\ddot{q}(t)\} + [\Phi]^T[C][\Phi]\{\dot{q}(t)\} + [\Phi]^T[K][\Phi]\{q(t)\} = [\Phi]^T\{F(t)\}$$

Define:

- $[M^*] = [\Phi]^T[M][\Phi]$
- $[C^*] = [\Phi]^T[C][\Phi]$
- $[K^*] = [\Phi]^T[K][\Phi]$
- $\{F^*(t)\} = [\Phi]^T\{F(t)\}$

If $[\Phi]$ is normalized such that:

$$[\Phi]^T[M][\Phi] = [I], \quad [\Phi]^T[K][\Phi] = [\Omega^2]$$

Then the modal equations become:

$$\{\ddot{q}(t)\} + [\Omega^2]\{q(t)\} = \{F^*(t)\}$$

If damping is neglected or proportional (Rayleigh damping), the damping matrix is also diagonalizable.

17.4 Orthogonality Conditions

The orthogonality properties of mode shapes are critical for decoupling. For undamped systems:

- **Mass Orthogonality**:

$$\phi_i^T [M] \phi_j = 0 \quad \text{for } i \neq j$$

- **Stiffness Orthogonality**:

$$\phi_i^T [K] \phi_j = 0 \quad \text{for } i \neq j$$

These conditions imply that the modal matrix diagonalizes both the mass and stiffness matrices, provided the system is classically damped or undamped.

17.5 Normalization of Mode Shapes

Mode shapes can be normalized with respect to mass:

$$\phi_i^T [M] \phi_i = 1$$

This simplifies the modal equations further:

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = F_i^*(t)$$

Where ω_i is the natural frequency of the ith mode.

17.6 Diagonalization of Matrices

- **Stiffness matrix [K]** and **mass matrix [M]** are symmetric and positive definite.
- Eigenvalue problem:

$$([K] - \lambda_i [M]) \phi_i = 0$$

- Eigenvectors ϕ_i are mutually orthogonal and can be used to construct the transformation matrix $[\Phi]$.

Diagonalization results in:

$$[\Phi]^T [M] [\Phi] = [I], \quad [\Phi]^T [K] [\Phi] = [\Omega^2]$$

Where $[\Omega^2] = \text{diag}(\omega_1^2, \omega_2^2, ..., \omega_n^2)$

17.7 Modal Superposition Method

Once decoupled, each scalar modal equation can be solved individually:

$$\ddot{q}_i(t) + 2\xi_i\omega_i\dot{q}_i(t) + \omega_i^2 q_i(t) = F_i^*(t)$$

Where ξ_i is the damping ratio of the ith mode.

The total response of the system is then obtained by summing modal responses:

$$\{u(t)\} = \sum_{i=1}^{n} \phi_i q_i(t)$$

This is called the **Modal Superposition Principle**.

17.8 Modal Truncation

In practice, not all modes contribute significantly to the response, especially in seismic analysis. Modal truncation involves retaining only the first few (usually 3–6) dominant modes, depending on their participation in the total response.

Criteria include:

- **Modal Mass Participation Factor**
- **Cumulative Effective Mass Ratio**

17.9 Special Case: Undamped Systems

For undamped systems, the modal equations are purely harmonic:

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = F_i^*(t)$$

These are second-order ODEs that can be solved using:

- Duhamel's Integral
- Convolution Integral
- Laplace Transform (for arbitrary loading)

17.10 Seismic Excitation: Base Acceleration Input

When the excitation is in the form of ground acceleration $\ddot{u}_g(t)$, the equations become:

$$[M]\{\ddot{u}(t)\} + [C]\{\dot{u}(t)\} + [K]\{u(t)\} = -[M]\{r\}\ddot{u}_g(t)$$

Where $\{r\}$ is the influence vector (usually a vector of ones).

In modal coordinates:

$$\ddot{q}_i(t) + 2\xi_i\omega_i\dot{q}_i(t) + \omega_i^2 q_i(t) = -\Gamma_i\ddot{u}_g(t)$$

Where $\Gamma_i = \phi_i^T[M]\{r\}$ is the **modal participation factor**.

17.11 Numerical Example (Optional for Students)

A simple 3-storey shear building with lumped masses and stiffness values can be analyzed to illustrate the decoupling process:

- Compute $[M]$, $[K]$
- Solve eigenvalue problem
- Normalize mode shapes
- Form modal transformation
- Decouple and solve modal equations
- Use modal superposition to obtain floor displacements

(Example omitted here but recommended for classroom or assignment.)

17.12 Modal Participation Factors

The **modal participation factor** Γ_i quantifies how much each mode contributes to the response of the system due to ground motion. It is calculated as:

$$\Gamma_i = \frac{\phi_i^T[M]\{r\}}{\phi_i^T[M]\phi_i}$$

Where:

- ϕ_i: i-th mode shape vector
- $[M]$: Mass matrix
- $\{r\}$: Influence vector (usually a vector of ones for ground motion in one direction)

This factor helps determine which modes are important for a particular loading condition. Larger participation factors indicate higher contribution of the mode to the total dynamic response.

17.13 Effective Modal Mass

The **effective modal mass** for the i-th mode is given by:

$$M_{\text{eff},i} = \Gamma_i^2 \cdot \phi_i^T [M] \phi_i$$

The **total effective mass** is used to check whether a sufficient number of modes have been included in modal analysis. Cumulative effective modal mass ratios are evaluated to determine if a sufficient portion (e.g., 90% or 95%) of the total mass has been accounted for.

17.14 Orthogonal Properties with Damping

In the presence of damping, decoupling depends on the type of damping used:

- For **classical (proportional) damping**, the modal matrix still diagonalizes the damping matrix.

$$[C] = \alpha[M] + \beta[K]$$

Where α and β are constants.

- For **non-classical damping**, the damping matrix is not diagonalizable by the modal matrix, and full decoupling may not be possible. In such cases, **complex modal analysis** or **state-space methods** are used.

17.15 Complex Modes and Non-Proportional Damping

When damping is **non-classical**, the system can exhibit **complex eigenvalues** and **complex mode shapes**. These are handled using state-space formulation:

$$\dot{X}(t) = AX(t) + B\ddot{u}_g(t)$$

Where $X(t)$ includes displacement and velocity, and A is the system matrix that includes damping effects.

In such systems, decoupling is not perfect, and modes may interact under seismic excitation. Approximate or numerical methods are used to analyze the response.

17.16 Coupling in Torsional and Asymmetric Systems

In real structures, particularly irregular buildings with asymmetry or torsional stiffness, complete decoupling may not be possible:

- **Plan irregularities** and **torsional modes** lead to coupling between translational and rotational DOFs.
- Modal analysis still applies but requires more modes and careful interpretation.
- Modal coupling may cause **torsional amplification**, especially in seismic-prone zones.

17.17 Use of Modal Analysis in Earthquake Response Spectra Method

In practical earthquake engineering design, modal analysis is used in the **response spectrum method**:

- Peak modal responses are computed using spectral acceleration values from a design spectrum.
- Modal responses are combined using methods such as:
  - **Square Root of Sum of Squares (SRSS)**
  - **Complete Quadratic Combination (CQC)**
- These techniques assume decoupled modes and rely on accurate decoupling.

17.18 Limitations of Modal Decoupling

While modal decoupling is a powerful technique, it has limitations:

- Assumes **linear elastic behavior**
- Assumes **classically damped systems**
- **Higher modes** may still influence local responses
- **Nonlinear behavior** under strong ground motion is not captured
- Does not address **duration**, **sequence**, or **directionality** of earthquakes directly

For nonlinear behavior or near-collapse conditions, **time-history analysis** or **nonlinear dynamic analysis** is used instead.
