

Hydraulic Engineering
Prof. Mohammad Saud Afzal
Department of Civil Engineering
Indian Institute of Technology – Kharagpur

Lecture – 20
Boundary Layer Theory (Contd..)

Welcome back to fourth lecture of the boundary layer analysis and we are going to start this lecture with Problem number 5.

(Refer Slide Time: 00:30)

Problem- 5

• The velocity profile for laminar boundary layer is given as: $\frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$. Find an expression for boundary layer thickness δ and the wall shear stress.



The question says that, the velocity profile for laminar boundary layer is given as u / U equal to $2y / \delta - (y / \delta)^2$. Now, we have to find an expression for boundary layer thickness δ and the wall shear stress.

(Refer Slide Time: 00:47)

Given $\frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$

$Q = \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$ → definition

$Q = \int_0^\delta \left(2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) \left(1 - 2\frac{y}{\delta} + \frac{y^2}{\delta^2}\right) dy$

$Q = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2} - \frac{4y^2}{\delta^2} + \frac{4y^3}{\delta^3} + \frac{y^4}{\delta^3} - \frac{y^5}{\delta^5}\right) dy$

$Q = \delta - \frac{\delta}{3} - \frac{4\delta}{3} + \frac{\delta}{5} + \delta$

$Q = \frac{2\delta}{15}$



So, what is given is, $u / U = 2 y / \delta - y / \delta$ whole square. Therefore, the momentum thickness can be written as, $\int_0^\delta u / U (1 - u / U) dy$. This is the definition of momentum thickness. So, it is going to be, $\int_0^\delta (2 y / \delta - y^2 / \delta^2) (1 - 2 y / \delta + y^2 / \delta^2) dy$, whole dy or the momentum thickness is going to be, $\int_0^\delta (2 y / \delta - y^2 / \delta^2 - 4 y^2 / \delta^2 + y^4 / \delta^3) dy$.

After integration and substituting the limits, the first term will give us, $\delta - \delta / 3 - 4 \delta / 3 - \delta / 5$ and the last term is give us δ . Therefore, θ is going to be $2 \delta / 15$. This is the momentum thickness. So, we proceed to the next page, where we are going to solve this question further. So, I am going to have a new.

(Refer Slide Time: 03:17)

Handwritten derivation on a whiteboard:

- Top left: $\theta = \frac{2\delta}{15}$ (boxed)
- Text: "From von-Karman momentum integral equation"
- Equation 1: $\tau_0 = \rho U^2 \frac{d\theta}{dx}$
 $= \rho U^2 \frac{d}{dx} \left(\frac{2\delta}{15} \right)$
 $\tau_0 = \frac{2}{15} \rho U^2 \frac{d\delta}{dx}$ (labeled ①)
- Equation 2: $\tau_0 = \mu \left(\frac{du}{dy} \right)_{y=0}$ (labeled ②)
- Text: "From Newton's law of viscosity"
- Equation 3: $\tau_0 = \mu \left(\frac{2U}{\delta} - \frac{2Uy}{\delta^2} \right)_{y=0}$
 $\tau_0 = \mu \frac{2U}{\delta}$ (labeled ③)
- Equation 4: "Equate ① and ②"
 $\frac{2}{15} \rho U^2 \frac{d\delta}{dx} = \frac{2\mu U}{\delta}$

So, we have got, θ is equal $2 \delta / 15$. Now, from von-Karman momentum integral equation, $\tau_0 = \rho U^2 d\theta / dx$. So, ρU^2 and we put the θ here. So, d of $2 \delta / 15$. Therefore, τ_0 is going to be $2 / 15 \rho U^2 d\delta / dx$ and this is equation number 1. Also τ_0 , we know, is can be written as $\mu du / dy$ at $y = 0$. And how do we get this? From Newton's law of viscosity.

So, velocity profile we already know. So, if u / U was given, in terms of y and δ . So, we can, so, from here, we can go τ_0 can be written as μ will be and now du / dy . So, it will become $2 U / \delta - 2 U / \delta^2$ into y at $y = 0$. So, this will become μ , this term will go away. When we substitute, it will be 2 , this is μ , so, $2 U / \delta$, this is equation number 2. Equate 1 and 2, $2 / 15 \rho U^2 d\delta / dx = 2 \mu U / \delta$.

So, this is the equation that we get. And now to find delta what we must do? We must simply integrate it. So, we will do this integration on the next page.

(Refer Slide Time: 06:52)

$$\delta d\delta = \frac{2\mu U}{15\rho U^2} dx$$

$$\delta d\delta = \frac{15\mu}{\rho U^2} dx$$

$$\delta d\delta = \frac{15\mu}{\rho U} \frac{dx}{U}$$

$$\int_0^\delta \delta d\delta = \int_0^x \frac{15\mu}{\rho U} \frac{dx}{U}$$

$$\Rightarrow \frac{\delta^2}{2} = \frac{15\mu}{\rho U} \frac{x}{U}$$

$$\Rightarrow \frac{\delta^2}{2} = \frac{15\mu}{\rho U^2} x$$

$$\Rightarrow \delta^2 = \frac{30\mu x^2}{\rho U}$$

$$\delta = \frac{5.48x}{\sqrt{Re_x}}$$

to obtain C → δ at leading edge is always zero
at leading edge $x=0, \delta=0 \Rightarrow C=0$

$\Rightarrow \delta^2 = \frac{15\mu}{\rho U} x$
 $\Rightarrow \frac{\delta^2}{2} = \frac{15\mu}{\rho U} \frac{x}{U}$

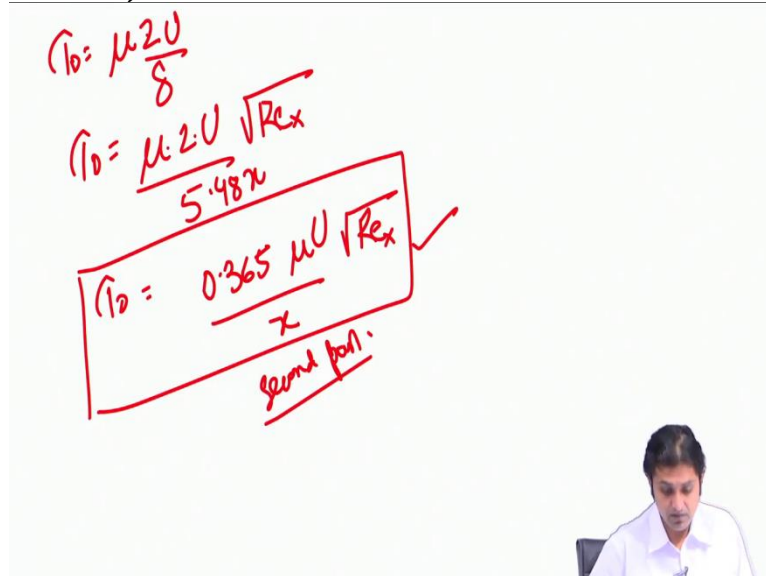
$\delta^2 = \frac{30\mu x^2}{\rho U}$
 $\delta = \frac{5.48x}{\sqrt{Re_x}}$
BL thickness

So, this integration will go like, $\delta d\delta = \frac{2\mu U}{15\rho U^2} dx$, you just rearrange the terms, divided by $2 / 15\rho U$ square dx . So, this will become $\delta d\delta$ is equal to, we can cancel this one, I mean, μU , when U will get cancelled. This is μ into $\rho U dx$, sorry, ρU square dx or $d\delta$, U , U get cancelled, so it will be $15\mu / \rho U dx$ and then we can integrate 0 to δ and this one will go from 0 to x .

So, 15μ divided by $\rho U dx$ or after integrating, it will become $\delta^2 / 2 = 15\mu$ divided by ρU , sorry, this is capital U , $x + C$. This is the C is constant of integration. But how do we obtain C ? We know to obtain C , we know that boundary layer thickness at leading edge is always 0. So, at leading edge, x is 0 and here δ is also 0. So, if you substitute this in this equation, we get $C = 0$, Therefore, we can simply say boundary layer thickness, $\delta^2 / 2 = 15\mu$ times $x / \rho U$.

Or just to simplify, what I am going to do? This left hand side will remain the same. So, if you divide μ / ρ , μ / ρ this becomes ν . And this U , multiply it with x numerator and denominator as well, and this one is written as Reynolds number at a distance x . So, I just conclude here, then δ^2 can be written as, we take 2 this side, that will be, $30x$ square divided by Re_x , or in other words, under root, so, 30 when done under root it becomes 5.48 at x divided by under root Re_x . So, this is our first answer, boundary layer thickness. Now, the second part, I will erase all ink.

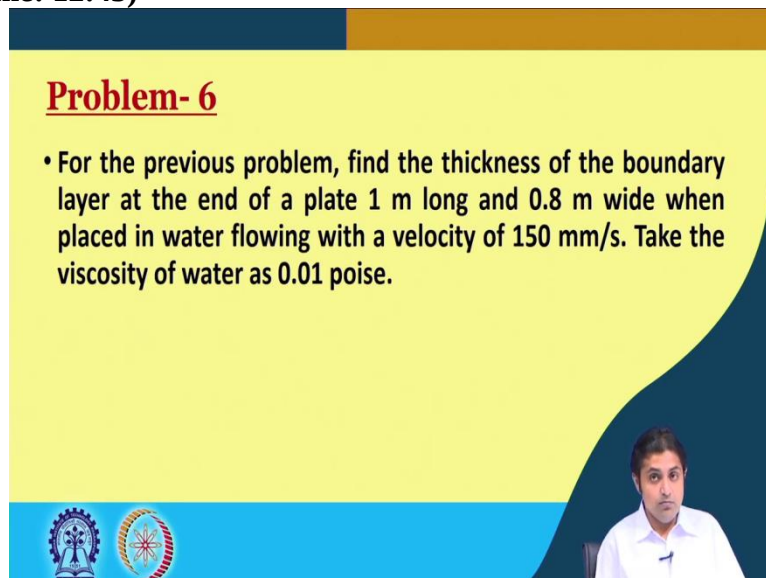
(Refer Slide Time: 11:27)



Handwritten notes on a whiteboard showing the derivation of boundary layer thickness formulas. The first formula is $\tau_0 = \frac{\mu 2U}{\delta}$. The second formula is $\tau_0 = \frac{\mu 2U \sqrt{Re_x}}{5.48x}$. The third formula, enclosed in a red box, is $\tau_0 = \frac{0.365 \mu U \sqrt{Re_x}}{x}$, with the text "second part." written below it.

So, for second, we know, that $\tau = \mu 2 U / \delta$ right, that is, what it came to. Now, it is a matter of just a substitution. So, τ_0 is $\mu 2 U \delta$ is $5.48 x \sqrt{Re_x}$. So, τ_0 comes to be $0.365 \mu U / x \sqrt{Re_x}$. So, after that is found you can express it in any form that you want. So, this is the second part. It was a long question, but actually clearly indicated how we can use the von-Karman momentum integral equation for finding the boundary layer thickness.

(Refer Slide Time: 12:43)



Problem- 6

- For the previous problem, find the thickness of the boundary layer at the end of a plate 1 m long and 0.8 m wide when placed in water flowing with a velocity of 150 mm/s. Take the viscosity of water as 0.01 poise.

The slide features a yellow background with a blue header and footer. The footer contains two circular logos on the left and a video inset of a man in a white shirt on the right.

So, now, the last problem in this set is that, for the previous problem find the thickness of the boundary layer at the end of plate 1 meter long and 0.8 meter wide when placed in water flowing with a velocity of 150 millimeters per second. So, the question, theoretically we have done. Now, we need to put in some values. And it says, take the viscosity of water as 0.01 poise. So, to solve this we will go to the white screen again.

(Refer Slide Time: 13:23)

Given: $L = 1\text{ m}$, $b = 0.8\text{ m}$
 $U = 150\text{ mm/s} = 150 \times 10^{-3}\text{ m/s}$
 $\mu = 0.01\text{ Poise} = 0.001\frac{\text{Ns}}{\text{m}^2}$
 from previous question
 $\delta = \frac{5.48 x}{\sqrt{Re_x}}$ (for laminar boundary layer)
 $x = 1\text{ m}$ (end of plate)
 $Re_x = \frac{Ux}{\nu} = \frac{\rho Ux}{\mu} = \frac{1000 \times 150 \times 10^{-3} \times 1}{0.001}$
 $Re_x = 1.5 \times 10^5$ ($< 5 \times 10^5 \Rightarrow \text{laminar}$)

As always, we will write given. So, length is given as 1 meter, b is given as 0.8 meter, U is 150 millimeters per second or we can write 150 into 10 to the power -3 meters per second, mu is given as 0.01 poise or in SI unit, we can write, 0.001 N second per meter square. So, what we have found out from the previous question?

Delta is $5.48 x$ divided by under root Re_x at x , Reynolds number for laminar boundary layer. Because why laminar? Because we have used $\tau = \mu \frac{du}{dy}$, that is, for the laminar boundary layer, At end of the plate $x = 1$ meter, end of plate. So, Reynolds number at this position will be Ux / ν or $\rho Ux / \mu$. So, 1000 is the density of the water, 150 into 10 to the power -3 into x is 1 meter divided by μ is 10 to the power -3.

So, Reynolds number at x is going to be 1.5 into 10 to the power 5. And of course, it is less than 5 into 10 to the power 5 means, laminar. So, our assumption was okay. So, I will, before proceeding.

(Refer Slide Time: 16:02)

$$\delta = \frac{5.48x}{\sqrt{Re_x}} = \frac{5.48 \times 1}{\sqrt{150000}} = 0.01415 \text{ m}$$

$$\boxed{\delta = 0.01415 \text{ m}}$$

Alright. So, delta by formula is 5.48 x divided by under root R e x. So, 5.48 into 1 divided by under root 150000 and that comes out to be 0.01415 meters. So, the real values, I mean, the numerical values come 0.01415 meters. So, this is the boundary layer thickness. Here, we have calculated the boundary layer thickness in terms of the numerical values.

(Refer Slide Time: 16:54)

Laminar Boundary Layer over a Flat Plate

- The analysis is done by applying von Karman Momentum Integral Equation.
- Assumptions:
 - u is a function of $\eta \left(= \frac{y}{\delta} \right)$ and invariant of x . Third order polynomial law
 - $\frac{u}{U} = f(\eta) = A + B\eta + C\eta^2 + D\eta^3$

So, we will now proceed to the other topic. So, we last, when we concluded before solving the problems what we said? We said was, that the equation 1, I mean, that equation is valid both for laminar and turbulent boundary layer. So, first we study laminar boundary layer over a flat plate. So, now, this analysis is done by applying von-Karman momentum integral equation, as it is common for both laminar and turbulent flow.

The assumption says that, u is a function of η . η is y / δ . y is the distance above the plate and δ is the boundary layer thickness and is not a function of x . So, u is not a function of x but only a function of y or to be precise y / δ . So, what we do is, we assume that, u / U is a function of η and this is written as $A + B \eta + C \eta^2 + D \eta^3$. So, how do we approach this problem? How many terms are there? 4 terms. How many unknowns? ABCD.

But fortunately, we know, that while doing those equations, I mean, von-Karman momentum integration equation, we had 4 boundary conditions, if you remember. I will just briefly, take you and show that. So, you see. So, what were those boundary conditions? Very simple.

(Refer Slide Time: 18:47)

Boundary Conditions

- $u(y = 0) = 0$ ✓
- $\frac{d^2u}{dy^2}(y = 0) = 0$ ✓
- $u(y = \delta) = U$ ✓
- $\frac{du}{dy}(y = \delta) = 0$ ✓

Boundary conditions. So, these were the 4 boundary conditions, if you remember. u at $y = 0$ is 0. $\frac{d^2u}{dy^2}$ at $y = 0$ is also 0. And u at $y = \delta = U$ and then $\frac{du}{dy}$ at the boundary layer, the velocity gradient at the top of the boundary layer is 0. There is a free stream velocity, these two are 0. So, now, we apply this condition for also for laminar cases. What we get? I mean, this function is a third order polynomial law.

So what we get? So, if you apply u at $y = 0$ is 0, which will give us $A = 0$. So, if you put, if $y = 0$ η is 0. So, $A + 0 + 0 + 0$ equals to 0. If you apply second boundary condition, $\frac{d^2u}{dy^2}$ at $y = 0$. So, you differentiate this f of η twice, so, this will become 0, this will become 0 on twice differentiation. This will give us $2C + 6D\eta = 0$. So, when $\eta = 0$, this means, $C = 0$.

(Refer Slide Time: 20:11)

- Applying the boundary conditions, we obtain

$$A = C = 0, B = \frac{3}{2} \text{ and } D = -\frac{1}{2}$$
- Therefore, we get

$$\frac{u}{U} = f(\eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3$$
- Inserting the above in von Karman Momentum integral equation:





So, the first 2 conditions are going to give us $A = C = 0$ and substituting third and fourth. So, fourth one, the third one where we have to differentiate once, will give us this condition, you can try this at home, and the fourth condition will give us D is equal to minus $1/2$. So, those 4 boundary conditions are used to solve for these value of A B C and D . Therefore, we can simply write that, u , I mean, u / U which is a free stream velocity is given by

$$\frac{u}{U} = f(\eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3$$

(Refer Slide Time: 21:00)

$$\tau_w = \rho U^2 \frac{d}{dx} \left[\int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U} \right) dy \right]$$

or

$$\tau_w = \rho U^2 \frac{d\delta}{dx} \int_0^1 f(1-f) d\eta$$

When y varies from 0 to δ ,
 η varies from 0 to 1

or

$$\tau_w = 0.139 \rho U^2 \frac{d\delta}{dx} \quad (\text{Eq. 11})$$

Handwritten notes on the right:

$$\eta = \frac{y}{\delta}$$

$$\Rightarrow y \rightarrow \eta \Rightarrow y = \eta \delta$$

$$\Rightarrow y = \delta \Rightarrow \eta = 1$$

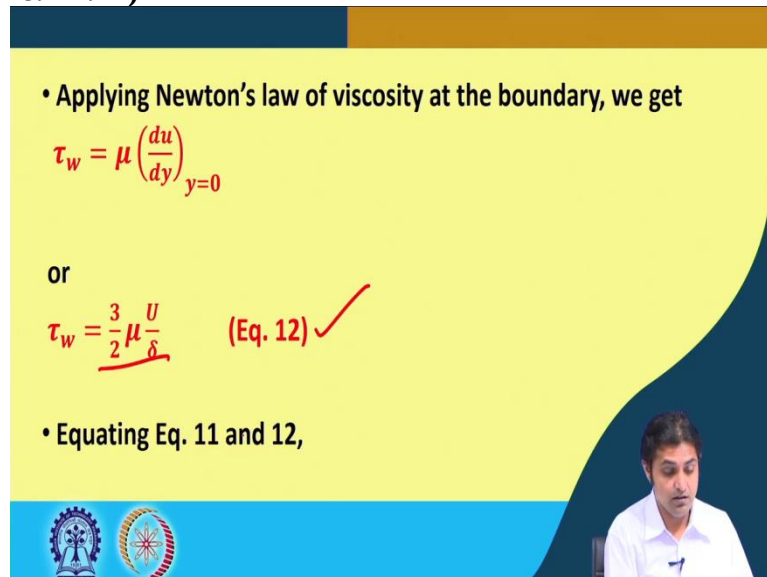



Now, if inserting the above in von Karman momentum integral equation. If we insert this, what do we get? So, τ_w is written as $\rho U^2 \frac{d\delta}{dx}$ of, this is momentum thickness, θ , or in other words, you see, this is $\int f(1-f) d\eta$. Because if, $\eta = y / \delta$, so, $d\eta = dy / \delta$

/ delta. So, now the limit will change because when y varies from 0 to delta eta can vary from 0 to 1. So, eta is y / delta. When y = 0 implies eta = 0 and when y = delta implies eta = 1.

So, this limit will change, as shown. Or in other terms, what we can write is, so, on integration and converting it into form of delta y / delta, this will give us tau w 0.139 rho U square remains, rho U square. So, this value comes out to be 0.139, you can see that here.

(Refer Slide Time: 22:44)



• Applying Newton's law of viscosity at the boundary, we get

$$\tau_w = \mu \left(\frac{du}{dy} \right)_{y=0}$$

or

$$\tau_w = \frac{3}{2} \mu \frac{U}{\delta} \quad (\text{Eq. 12}) \checkmark$$

• Equating Eq. 11 and 12,

Now, if we are apply the Newton's law of viscosity at the boundary. Why can we apply the Newton's law of viscosity? Because it is a laminar flow. We did the same principle when we were solving the numerical problems. Or, if you see, du / dy at y = 0. So, you see, so, du / dy at y = 0, it is going to give us 3 / 2 mu into U / delta.

So, what do we do next? One way we have obtain Newton law viscosity and the other equation was, equation number 11 here, because what we started with was the velocity profile here, u / U = A+ B eta + C eta square. And then differentiating that, we got this and we are going to equate equation 11 and equation 12.

(Refer Slide Time: 23:55)

$$\delta d\delta = 10.79 \frac{\nu}{U} dx$$

- Integrating the above equation

$$\int_0^\delta \delta d\delta = 10.79 \frac{\nu}{U} \int_0^x dx$$

or

$$\frac{\delta^2}{2} = 10.79 \frac{\nu}{U} x \quad \text{or} \quad \frac{\delta^2}{2} = 10.79 \frac{\nu}{U} x^2$$

$\frac{1}{Re_x}$

This will give us, delta d delta is equal to 10.79 $\nu / U dx$. And if we integrate this above equation, we can get the boundary layer thickness. Actually, the last problem that was their problem number 5 and 6 was exactly the same problem that we solved the way, I mean, the question was different, but the way of solving was the same as this derivation. So, we can write, delta square = 10.79 $\nu / U x$ or delta square = 10.79 $\nu / U x$.

So, what we did was, we multiplied x and x up stair, I mean, denominator here also here also. So, this becomes x square and this is Reynolds number. This is what exactly we did.

(Refer Slide Time: 24:51)

or

$$\delta = \frac{4.64x}{\sqrt{Re_x}}$$

- The above equation can be utilized for obtaining the expressions of τ_w and F_{drag} .

From the expression of Newton's Law of Viscosity applied at the wall.

$$F_{drag} = \int_0^L \tau_w dx$$

Or in other words, we get, delta = 4.64 x under root Reynolds number of x. Try to remember the question that we did, just finished in the last, I mean, at the beginning of this lecture. So, the above equation can be utilized for obtaining expressions for tau w and F drag because we

get the boundary layer thickness. And from the expression of Newton's law of viscosity, we can obtain τ_w . And F_{drag} , we can simply use. How? That we integrate, τ_w from 0 to over the entire plate. So, this is the way that it is done.

(Refer Slide Time: 25:39)

Problem- 7

• For the velocity profile for laminar boundary layer $\frac{u}{U} = \frac{3}{2} \left(\frac{y}{\delta} \right) - \frac{1}{2} \left(\frac{y}{\delta} \right)^3$, determine the shear stress and the drag force in terms of Reynolds number.



Now, we will solve 1 problem. The problem is, for the velocity profile for laminar boundary layer $u / U = 3 / 2 y / \delta - \text{half } y / \delta \text{ to the whole cube}$, determine the shear stress and the drag force in terms of Reynolds number. We have done a similar problem but we will do another one now, to make it more clear.

(Refer Slide Time: 26:06)

Given

$$\frac{u}{U} = \frac{3}{2} \left(\frac{y}{\delta} \right) - \frac{1}{2} \left(\frac{y}{\delta} \right)^3$$

For the given velocity profile, we have seen

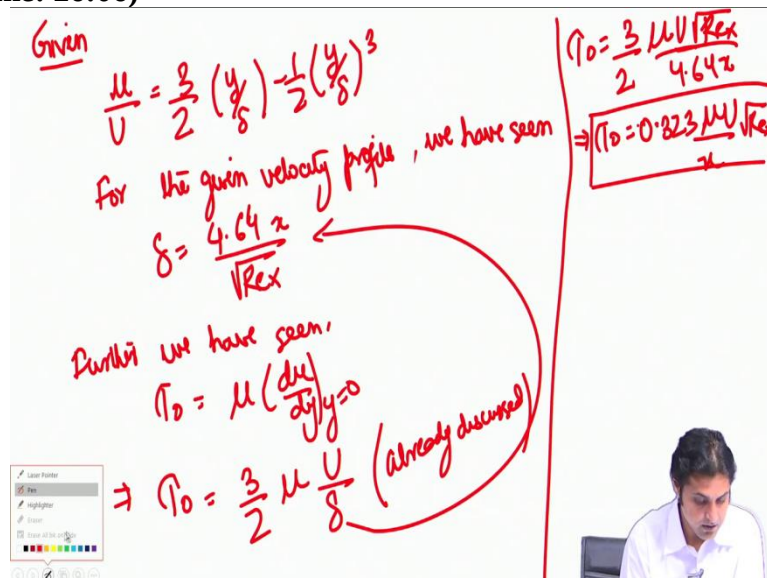
$$\delta = \frac{4.64 x}{\sqrt{Re_x}}$$

Further we have seen,

$$\tau_0 = \mu \left(\frac{du}{dy} \right)_{y=0}$$

$$\Rightarrow \tau_0 = \frac{3}{2} \mu \frac{U}{\delta} \quad (\text{already discussed})$$

$$\tau_0 = \frac{3}{2} \frac{\mu U \sqrt{Re_x}}{4.64 x}$$

$$\Rightarrow \tau_0 = 0.823 \frac{\mu U \sqrt{Re_x}}{x}$$


So, first write what is given to us. So, $u / U = 3 / 2 y / \delta - 1 / 2 y / \delta \text{ to the whole cube}$. For the given velocity profile, we have seen that δ came out to be $4.64 x \text{ under root}$

Reynolds number at x . This was the last derivation that we saw. Further, we have seen, $\tau_0 = \mu \frac{du}{dy}$ at $y = 0$ implies $\tau_0 = \frac{3}{2} \mu U \sqrt{\frac{Re}{x}}$. This has already been discussed.

So, δ , we know, from here. So, $\tau_0 = \frac{3}{2} \mu U \sqrt{\frac{Re}{x}}$. Or finally, τ_0 can be written as $0.323 \mu U \sqrt{\frac{Re}{x}}$. Now, the second part is, how do we find the drag force? Very simple, we have laid down the procedure.

(Refer Slide Time: 28:45)

Drag force L
 $F_D = \int_0^L \tau_0 b dx$
 $\Rightarrow F_D = \int_0^L 0.323 \frac{\mu U}{x} \sqrt{Re x} b dx$
 $\Rightarrow F_D = \int_0^L \frac{0.323 \mu U b}{x} \left(\frac{\rho U x}{\mu} \right)^{1/2} dx$
 $\Rightarrow F_D = \frac{0.323 \mu U b \rho^{1/2} U^{1/2}}{\mu^{1/2}} \int_0^L x^{-1/2} dx$
 $\Rightarrow F_D = 0.323 \times 2 \times \mu U b \sqrt{\frac{\rho U L}{\mu}}$

$L \rightarrow$ length over which the drag is intended to be found.
 $x \rightarrow 0$ to L

So, drag force F_D , can be found out using integral 0 to L $\tau_0 b dx$ implies drag force $F_D = \int_0^L 0.323 \mu U \sqrt{\frac{Re}{x}} b dx$. Here, L is the length over which the drag is intended to be found. So, implies, drag force integral 0 to L $0.323 \mu U b \sqrt{\frac{\rho U x}{\mu}}$ to the power $1/2 dx$. So, implies, F_D actually can be written as, after integration $0.323 \mu U b \rho^{1/2}$ it will come out, $U^{1/2}$ it will come out, $\mu^{1/2}$ it will come out integral 0 to L $x^{-1/2} dx$.

So, after integrating and substituting the values of x from 0 to 1, 0 to L x goes from 0 to L . F_D will be $0.323 \times 2 \times \mu U b \sqrt{\frac{\rho U L}{\mu}}$. So, this is the drag force. More importantly, it is this way that it has to be found, I mean, in your questions you will be mostly given an objective question. If you know the way how to find it out, I mean, it will not be so, tedious calculation.

So, I would like to end the lecture at this point in time and we will go through the some more concepts and some more questions in our next class. Thank you so much.