Chapter 10: Duhamel Integral

Introduction

In the study of earthquake engineering, the dynamic response of structures to ground motion is a fundamental aspect. The **Duhamel Integral** provides a mathematical formulation for determining the response of a linear time-invariant (LTI) single-degree-of-freedom (SDOF) system to arbitrary dynamic loading. When the loading varies with time—such as in the case of an earthquake—analytical solutions to the equations of motion become complex. Duhamel's integral is a powerful tool that allows us to express the response of the system in terms of a convolution integral of the excitation with the system's impulse response function.

This chapter explores the theoretical background, derivation, and application of the Duhamel Integral in the context of structural dynamics and earthquake response analysis.

10.1 Equation of Motion for Linear SDOF System

Consider a single-degree-of-freedom (SDOF) system subjected to an external time-varying force $F(t)$. The general equation of motion is given by:

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F(t)$$

Where:

- m = mass of the system
- c = damping coefficient
- k = stiffness of the system
- $x(t)$ = displacement as a function of time
- $F(t)$ = applied external force

This second-order linear differential equation with constant coefficients governs the motion of the system under arbitrary loading.

10.2 Impulse Response Function

Before deriving the Duhamel Integral, we define the **impulse response function** $h(t)$ as the response of the system to a unit impulse force applied at $t=0$:

$$F(t)=\delta(t)$$

The solution $x(t)$ for such an input is the **unit impulse response**, which depends on the damping condition of the system:

10.2.1 Underdamped Case $(\zeta<1)$

Let $\omega_n=\sqrt{k/m}$ be the natural circular frequency and $\zeta=\frac{c}{2\sqrt{km}}$ be the damping ratio.

The impulse response function is given by:

$$h(t)=\frac{1}{m\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t)$$

Where $\omega_d=\omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency.

10.3 Derivation of Duhamel's Integral

The system's response to a general force $F(t)$ can be obtained using the principle of **superposition** and **convolution**. The idea is that any arbitrary force can be broken down into infinitesimally small impulses over time.

Using the superposition of the effects of these impulses, the total response is given by:

$$x(t)=\int_0^t h(t-\tau)F(\tau)\,d\tau$$

This is the **Duhamel Integral**, where:

- $x(t)$ is the displacement response at time t
- $h(t-\tau)$ is the impulse response function
- $F(\tau)$ is the force at time τ
- τ is a dummy time variable

10.4 Physical Interpretation

Duhamel's integral expresses the total system response as the weighted accumulation of the impulse responses over time. Each infinitesimal force $F(\tau)d\tau$ applied at an earlier time τ causes a delayed response that persists until time t, and the integral sums up these effects.

This is particularly useful in earthquake engineering, where the ground motion is often represented as a time-varying force or acceleration.

10.5 Application to Base Excitation (Earthquake Ground Motion)

In earthquake engineering, instead of an external force $F(t)$, the system is subjected to **ground acceleration** $\ddot{u}_g(t)$, which induces relative motion in the structure.

The equation of motion in this case becomes:

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=-m\ddot{u}_g(t)$$

This is a base-excited SDOF system, where $\ddot{u}_g(t)$ is the ground acceleration input from an earthquake. Let's denote:

$$F(t)=-m\ddot{u}_g(t)$$

Then using Duhamel's integral:

$$x(t)=-\int_0^t h(t-\tau)m\ddot{u}_g(\tau)d\tau$$

Or:

$$x(t)=-\frac{1}{\omega_d}\int_0^t e^{-\zeta\omega_n(t-\tau)}\sin[\omega_d(t-\tau)]\ddot{u}_g(\tau)d\tau$$

This provides a mathematical formulation to compute the **relative displacement response** of a structure to earthquake ground motion.

10.6 Numerical Evaluation of Duhamel Integral

In practice, earthquake records are available in digital form, so the integral must be evaluated numerically. Numerical techniques like:

- Trapezoidal Rule
- Simpson's Rule
- Step-by-step integration (e.g., Newmark's method)

are used to evaluate the integral over discrete time intervals.

Let Δt be the time step, and the time history be sampled at $t_i = i\,\Delta t$, then:

$$x(t_i) \approx -\sum_{j=0}^{i} \frac{1}{\omega_d} e^{-\zeta \omega_n (t_i - t_j)} \sin[\omega_d (t_i - t_j)]\, \ddot{u}_g(t_j)\, \Delta t$$

This approach is used to generate **response spectra** and time-history plots for structural analysis.

10.7 Duhamel's Integral for Zero Initial Conditions

The derivation assumes zero initial displacement and velocity:

$$x(0) = 0, \dot{x}(0) = 0$$

For non-zero initial conditions, an additional **homogeneous solution** must be added, which accounts for the free vibration response of the system. In earthquake engineering, however, structures are generally assumed to be at rest before the earthquake, so this assumption holds true in most cases.

10.8 Advantages and Limitations

Advantages:

- Provides an exact analytical solution for linear systems under arbitrary forcing.
- Fundamental to the formulation of response spectrum analysis.
- Applicable to both force and base excitation problems.

Limitations:

- Valid only for **linear** and **time-invariant** systems.

- Requires knowledge of the impulse response function, which may be difficult for complex systems.
- Not suitable for **nonlinear** systems or systems with time-varying properties.

10.9 Extension to Multi-Degree-of-Freedom (MDOF) Systems

While Duhamel’s integral is most commonly applied to SDOF systems, it can be extended to linear MDOF systems using **modal analysis**. Each mode is treated as a separate SDOF system, and their individual responses (via Duhamel's integral) are superimposed to find the total system response.

$$x(t)=\sum_{r=1}^{n} \phi_r q_r(t)$$

Where:

- ϕ_r = mode shape
- $q_r(t)$ = modal coordinate obtained using Duhamel's integral for each mode

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10.10 Convolution Integral and System Linearity

The Duhamel integral is a direct application of the **convolution integral**, which is valid only for **linear time-invariant (LTI)** systems. The mathematical form of convolution:

$$x(t)=\int_0^t h(t-\tau)F(\tau)\,d\tau$$

implies that the system output is the convolution of the **impulse response** $h(t)$ and the **input function** $F(t)$. The principle of superposition must hold, which is why this integral is not suitable for **nonlinear** systems.

This foundational idea is also critical in structural control, digital signal processing, and filtering in vibration analysis.

10.11 Alternative Representation using Convolution Theorem (Laplace Domain)

Using **Laplace Transforms**, the convolution in time domain corresponds to multiplication in the Laplace domain:

Let:

- $X(s)$ = Laplace Transform of displacement $x(t)$
- $H(s)$ = Laplace Transform of impulse response $h(t)$
- $F(s)$ = Laplace Transform of input force $F(t)$

Then:

$$X(s) = H(s) \cdot F(s)$$

Taking the inverse Laplace transform gives $x(t)$, the time-domain response. This method is especially useful for solving problems with known Laplace pairs and handling complicated initial conditions.

10.12 Response of Systems with Different Damping Levels

The system response via Duhamel's integral changes significantly based on the damping ratio ζ. Three cases are considered:

10.12.1 Underdamped System $(\zeta < 1)$

Oscillatory decay with impulse response function involving sine terms.

10.12.2 Critically Damped System $(\zeta = 1)$

Impulse response:

$$h(t) = \frac{t}{m} e^{-\omega_n t}$$

System returns to equilibrium fastest without oscillating.

10.12.3 Overdamped System $(\zeta > 1)$

Impulse response includes two exponential terms with no oscillation. The system returns to equilibrium slowly.

Each damping case affects the integral formulation and the transient behavior of the structure differently.

10.13 Energy Dissipation and Duhamel Response

The **energy dissipated** due to damping during vibration can be indirectly evaluated using the Duhamel response. For any time-dependent force $F(t)$, the **instantaneous power** transferred into the system is:

$$P(t)=F(t)\cdot\dot{x}(t)$$

Integrating this power over time provides the total work done on the system, part of which is dissipated through damping. This analysis is helpful in estimating how much of earthquake energy is absorbed by damping devices or structural elements.

10.14 Practical Application: Earthquake Ground Motion Records

When applying Duhamel's integral to real-life earthquake ground motion, the input $\ddot{u}_g(t)$ is obtained from seismographs in discrete time form (acceleration vs. time). The structural engineer uses this data to compute:

- Displacement $x(t)$
- Velocity $\dot{x}(t)$
- Acceleration $\ddot{x}(t)$

via numerical evaluation of the integral. The peak responses are used in **design spectra** and **seismic qualification**.

10.15 Programming Implementation (MATLAB/Python)

The Duhamel integral is often implemented computationally. A typical implementation involves:

- Discretizing the time vector.
- Using the known $\ddot{u}_g(t)$ or $F(t)$ values.

- Calculating $h(t)$ at each time step.
- Applying numerical integration (e.g., `numpy.convolve` in Python or `conv` in MATLAB).

Sample code is often written to generate **time-history plots**, **peak displacement**, and **response spectra**.

10.16 Limitations in Earthquake Engineering Practice

While powerful, Duhamel's integral has the following limitations:

- Assumes **linearity** of structure (no material or geometric nonlinearity).
- Not suitable for **inelastic behavior**—plastic hinges, yielding, cracking, etc.
- Requires **zero initial conditions** for simplified derivation.
- Requires extensive computation for real, multi-degree-of-freedom buildings unless modal superposition is applied.

For nonlinear systems, **time-history integration methods** like Newmark-beta or Wilson-θ are preferred.
