Chapter 16: Introduction to MDOF Systems

16.1 Introduction

In real-world structures such as buildings, bridges, and towers, the dynamic response to ground motion or external forces cannot be accurately captured using single-degree-of-freedom (SDOF) models alone. These structures have multiple interconnected components (floors, frames, columns, etc.), each of which can move independently in response to dynamic loading. Hence, **Multi-Degree-of-Freedom (MDOF)** models are essential for realistic dynamic analysis, especially in **Earthquake Engineering**, where lateral loads and seismic excitations act simultaneously on different points of a structure.

MDOF systems represent structures with more than one coordinate (degree of freedom) required to define their motion. This chapter introduces the formulation, characteristics, and behavior of MDOF systems, particularly in the context of linear elastic analysis under dynamic loading.

16.2 Degrees of Freedom in Structural Systems

A **degree of freedom (DOF)** is the number of independent displacements or rotations needed to define the configuration of a structure. In MDOF systems:

- Each floor of a building typically represents one DOF in a lumped mass model (translational motion).
- In frame structures, rotational DOFs may also be considered.
- A structure with n floors would typically have n translational degrees of freedom in planar analysis.

16.3 Mathematical Modeling of MDOF Systems

16.3.1 Lumped Mass Idealization

In seismic analysis, mass is usually lumped at each floor level (nodes), and the stiffness is represented by springs connecting these nodes. The model simplifies:

- Mass matrix $[M]$: Diagonal matrix with masses at each DOF.
- Stiffness matrix $[K]$: Symmetric matrix representing inter-storey stiffness.
- Damping matrix $[C]$: Often assumed as proportional damping for simplicity.

16.3.2 Equations of Motion

For an undamped, linear elastic MDOF system subjected to external forces:

$$[M]\{\ddot{u}(t)\}+[K]\{u(t)\}=\{f(t)\}$$

For a damped system:

$$[M]\{\ddot{u}(t)\}+[C]\{\dot{u}(t)\}+[K]\{u(t)\}=\{f(t)\}$$

Where:

- $\{u(t)\}$ = displacement vector
- $\{\dot{u}(t)\}$ = velocity vector
- $\{\ddot{u}(t)\}$ = acceleration vector
- $\{f(t)\}$ = external force vector

16.4 Free Vibration of Undamped MDOF Systems

16.4.1 Eigenvalue Problem

Solving the free vibration problem (no damping, no external force) gives:

$$[M]\{\ddot{u}(t)\}+[K]\{u(t)\}=0$$

Assuming a harmonic solution $\{u(t)\}=\{\phi\}\sin(\omega t)$, we get:

$$([K]-\omega^2[M])\{\phi\}=0$$

This leads to the **eigenvalue problem**, where:

- ω^2 are the **eigenvalues** (squared natural frequencies)
- $\{\phi\}$ are the **mode shapes** (eigenvectors)

The system has *n* natural frequencies and mode shapes for *n* DOFs.

16.4.2 Orthogonality of Mode Shapes

Mode shapes are **orthogonal** with respect to mass and stiffness matrices:

$$\{\phi_i\}^T[M]\{\phi_j\}=0 \text{ for } i\neq j$$

$$\{\phi_i\}^T [K]\{\phi_j\} = 0 \text{ for } i \neq j$$

This orthogonality allows **modal decoupling**.

16.5 Modal Analysis and Modal Superposition

Modal analysis transforms the coupled MDOF system into uncoupled SDOF systems using the **modal matrix** $[\Phi]$:

$$[M] = ¿$$

Each mode behaves like a separate SDOF system, and the total response is obtained using **modal superposition**:

$$\{u(t)\} = \sum_{i=1}^{n} ¿ \phi_i\} q_i(t) ¿$$

Where:

- $\{\phi_i\}$: Mode shape of the i-th mode
- $q_i(t)$: Generalized coordinate (modal response)

16.6 Damped MDOF Systems

16.6.1 Classical (Proportional) Damping

Proportional damping assumes:

$$[C] = \alpha [M] + \beta [K]$$

Where α and β are constants. This assumption ensures that modal decoupling is still valid and results in *classical damping*.

16.6.2 Non-Classical Damping

In reality, damping is often **non-classical**, where modal decoupling is not possible. Advanced numerical techniques or state-space methods are required in such cases.

16.7 Seismic Excitation in MDOF Systems

For base excitation (earthquake input):

$$[M]\{\ddot{u}(t)\}+[C]\{\dot{u}(t)\}+[K]\{u(t)\}=-[M]\{r\}\ddot{u}_g(t)$$

Where:

- $\{r\}$: Influence vector (typically all 1s for uniform ground motion)
- $\ddot{u}_g(t)$: Ground acceleration

Modal analysis still applies, transforming ground motion input into modal coordinates.

16.8 Numerical Solution Methods for MDOF Systems

Analytical solutions are feasible only for small systems. For real structures, numerical integration is essential.

16.8.1 Time Integration Methods

- **Newmark-beta Method**: Widely used, unconditionally stable for certain parameters.
- **Wilson-θ Method**: Stable for large time steps.
- **Runge-Kutta Methods**: Explicit, conditionally stable.

16.8.2 Modal Truncation

Only a few dominant modes (typically the first 3–5) contribute significantly to dynamic response, especially for seismic analysis. This helps reduce computational effort.

16.9 Practical Aspects in Structural Modeling

- **Mass distribution**: Accurate mass modeling (including non-structural elements) is essential.
- **Stiffness estimation**: May be obtained from member properties or stiffness matrices of finite elements.
- **Boundary conditions**: Proper constraints must be modeled (e.g., fixed base, soil-structure interaction).

- **Damping estimation**: Usually obtained from experimental data or empirical assumptions.

16.10 Applications in Earthquake Engineering

MDOF systems are used in:

- **Dynamic analysis of buildings**: Multi-storey frame or shear models
- **Seismic design**: Modal response spectrum analysis, time history analysis
- **Retrofit design**: Evaluating impact of base isolation, damping devices
- **Bridge analysis**: Long-span and irregular bridge structures

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16.11 Modal Response Spectrum Analysis

Response Spectrum Method is a key tool in earthquake engineering used to estimate the maximum response of MDOF systems subjected to seismic loading.

16.11.1 Concept

- It uses **response spectra** (graphs of peak response vs. natural period) developed from ground motion records.
- Instead of applying time-history ground motion, it estimates the peak response for each mode from the response spectrum.

16.11.2 Steps in Modal Response Spectrum Analysis

1. Perform **modal analysis** to obtain mode shapes and natural frequencies.

2. Calculate **modal participation factors**:

$$\Gamma_i = \frac{\{\phi_i\}^T [M] \{r\}}{\{\phi_i\}^T [M] \{\phi_i\}}$$

3. Compute **modal masses** and **effective participation**.

4. Obtain spectral acceleration $S_{a,i}$ from the response spectrum for each mode.

5. Compute peak modal response:

$$u_i = \phi_i \Gamma_i S_{a,i}$$

6. Combine modal responses using **modal combination rules**:

- Square Root of Sum of Squares (**SRSS**)
- Complete Quadratic Combination (**CQC**) for closely spaced modes

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16.12 Base Isolation and Its Modeling in MDOF Systems

Base isolation is a seismic protection technique used to reduce inter-storey forces and displacements in buildings.

16.12.1 Concept of Base Isolation

- Decouples the superstructure from ground motion.
- Uses isolators (rubber bearings, friction pendulum systems) between foundation and superstructure.

16.12.2 Modeling in MDOF Systems

- Add one additional DOF for base movement.
- Isolation system is modeled with stiffness K_b and damping C_b.
- Modified mass and stiffness matrices are used.
- Significant change in **mode shapes**—first mode becomes dominant and includes base displacement.

16.13 Torsional Effects in MDOF Systems

16.13.1 Introduction

In unsymmetrical buildings, the **center of mass (CM)** and **center of stiffness (CS)** do not coincide, resulting in **torsional coupling**.

16.13.2 Effects and Modeling

- Torsional motion leads to increased demands on edge columns and frames.
- Requires at least **3 DOFs per floor** in planar models: two translational (X and Y) and one rotational (θ).
- Offsets in mass and stiffness matrices create **coupled equations of motion**.

16.14 Numerical Example: 2-DOF System

A typical worked-out numerical example includes:

- Mass matrix:

$$[M]=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

- Stiffness matrix:

$$[K]=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

- Solving for:
    - Natural frequencies
    - Mode shapes
    - Modal participation
    - Response to ground motion (using modal superposition or response spectrum)

16.15 Use of Software Tools for MDOF Analysis

Modern structural analysis for MDOF systems is performed using tools like:

- **ETABS**, **SAP2000**, **STAAD Pro**, **ANSYS**, etc.
- These tools perform:
    - Modal analysis
    - Response spectrum analysis
    - Time-history analysis
    - Pushover analysis (for nonlinear systems)

Engineers must understand MDOF theory to correctly interpret the software output.

16.16 Limitations of Linear MDOF Models

- Assumes **linearity**, which is not valid for severe ground motions.
- Ignores **material and geometric nonlinearities**.
- Real structures have **non-classical damping** and **interaction with soil**, not captured in basic MDOF models.
- Requires **nonlinear time history analysis** or **pushover analysis** for performance-based design.