Chapter 15: Mode Shapes

Introduction

In the context of earthquake engineering and structural dynamics, **mode shapes** represent the characteristic deformation patterns that structures undergo at specific natural frequencies during free vibration. These shapes provide vital insights into how structures respond dynamically and are central to modal analysis — a core concept in evaluating seismic behavior. Understanding mode shapes is crucial for designing earthquake-resistant structures and ensuring that resonance or amplified motions do not result in catastrophic failure.

In this chapter, we will explore how mode shapes are derived, interpreted, and applied to civil engineering structures. We also delve into orthogonality conditions, normalization techniques, and their significance in response spectrum and time history analyses.

15.1 Free Vibration and Mode Shapes

Free vibration refers to the vibration of a system without any external force, after an initial disturbance. In multi-degree-of-freedom (MDOF) systems, the vibration response is a combination of several independent **modes of vibration**, each with a unique **natural frequency** and **mode shape**.

- **Definition**: A mode shape is the deformation pattern of a structure at a specific natural frequency during free vibration.
- Each mode shape corresponds to one natural frequency.
- The number of possible mode shapes is equal to the number of degrees of freedom (DOFs) in the system.

Example:

For a 3-DOF shear building, three distinct mode shapes will exist, each representing a different vertical displacement profile across the floors.

15.2 Mathematical Formulation of Mode Shapes

Consider an undamped linear MDOF system governed by:

$$[M]\{\ddot{u}\} + [K]\{u\} = \{0\}$$

Where:

- $[M]$ = Mass matrix
- $[K]$ = Stiffness matrix
- $\{u\}$ = Displacement vector

Assuming a harmonic solution of the form:

$$\{u(t)\} = \{\phi\} \sin(\omega t)$$

Substituting and simplifying leads to the **eigenvalue problem**:

$$([K] - \omega^2[M])\{\phi\} = \{0\}$$

- ω = natural frequency
- $\{\phi\}$ = mode shape (eigenvector)

This formulation yields **eigenvalues** (natural frequencies squared) and **eigenvectors** (mode shapes).

15.3 Properties of Mode Shapes

15.3.1 Orthogonality of Mode Shapes

Mode shapes are **orthogonal** with respect to both the mass and stiffness matrices:

- **Mass orthogonality**:

$$\{\phi_i\}^T[M]\{\phi_j\} = 0 \quad \text{for} \quad i \neq j$$

- **Stiffness orthogonality**:

$$\{\phi_i\}^T[K]\{\phi_j\} = 0 \quad \text{for} \quad i \neq j$$

Orthogonality is key in **modal superposition** and **decoupling** the equations of motion.

15.3.2 Normalization of Mode Shapes

Mode shapes are not unique in magnitude. They can be **normalized** for analytical convenience:

- **Mass normalization**:

$$\{\phi\}^T[M]\{\phi\} = 1$$

- **Stiffness normalization**:

$$\{\phi\}^T[K]\{\phi\} = \omega^2$$

15.4 Computation of Mode Shapes

Mode shapes can be computed by solving the eigenvalue problem using:

- **Analytical methods** for small systems (e.g., 2-DOF, 3-DOF)
- **Numerical methods** for large systems using:
  - Subspace iteration
  - Lanczos algorithm
  - Rayleigh-Ritz method

For high-rise or complex structures, software like SAP2000, ETABS, or ANSYS is commonly used.

15.5 Interpretation of Mode Shapes in Structural Dynamics

15.5.1 First Mode Shape

- Usually involves **global movement** of the entire structure.
- Dominant in seismic analysis due to its **lower frequency** and **higher participation**.

15.5.2 Higher Mode Shapes

- Represent **localized or complex motion**.
- Become significant in **irregular** or **tall structures**.
- Often show **curvatures**, **torsions**, or **out-of-phase displacements** between different parts.

15.6 Mode Shapes of Typical Structures

15.6.1 Shear Building

- Floors are modeled as lumped masses.
- Columns are assumed to provide lateral stiffness.
- Mode shapes usually depict lateral deflection with increasing curvature in higher modes.

15.6.2 Cantilever Beam

- Mode shapes resemble sine waveforms.
- 1st mode: single curvature
- 2nd mode: double curvature, and so on.

15.6.3 Frame Structures

- May exhibit lateral translation, torsion, and combined mode shapes.
- Torsional mode shapes are critical in asymmetrical buildings.

15.7 Significance in Earthquake Engineering

- **Seismic Response**: Structures tend to vibrate in their natural modes during seismic excitation.
- **Design Optimization**: Identifying mode shapes helps in modifying geometry or stiffness to improve performance.
- **Modal Participation Factor**: Indicates how much each mode contributes to the overall response.
- **Time History and Response Spectrum Analysis**: Use mode shapes to determine maximum displacements and internal forces.
- **Mode Combination Rules**: Use of SRSS (Square Root of Sum of Squares) or CQC (Complete Quadratic Combination) for combining modal responses.

15.8 Experimental Determination of Mode Shapes

Mode shapes can also be found through:

- **Ambient Vibration Testing**
- **Shake Table Testing**
- **Impact Hammer Testing**

Data is recorded using accelerometers or laser vibrometers, and processed using modal analysis software.

15.9 Influence of Mass and Stiffness Distribution

- Uneven mass or stiffness can result in **mode localization** and **torsional modes**.
- Symmetrical distribution generally yields well-separated and simpler mode shapes.

- Irregularities must be carefully modeled to capture correct mode shapes for design.

15.10 Use in Structural Control and Retrofitting

- Mode shapes help identify **weak stories**, **soft floors**, and **critical joints**.
- Used in design of **tuned mass dampers (TMDs)**, **base isolation systems**, and **retrofit schemes**.
- Changes in mode shapes before and after retrofitting provide insight into structural improvement.
