Solid Mechanics
Prof. Ajeet Kumar
Deptt. of Applied Mechanics
IIT, Delhi
Lecture - 16
Stress-Strain relation for Isotropic Materials

Hello everyone! Welcome to lecture 16! In this lecture, we will learn about a specific case of linear stress-strain relation, i.e., isotropic stress-strain relation.

1 Isotropic Materials (start time: 01:12)

We had learnt the following general form of linear stress-strain relation:

$$\sigma_{ij} = \sum_{k,l} C_{ijkl}\ \epsilon_{kl} \tag{1}$$

The stiffness tensor C_{ijkl} has 21 independent constants for a general material. Such materials are also called anisotropic materials which means that at a point in the body, the material properties are different in different directions. We will now discuss about a special material called isotropic materials which have the same property in every direction. To understand this, consider a body made up of an isotropic material. We stretch the material at an arbitrary point along $\underline{e}_1$ direction as shown in Figure 1.

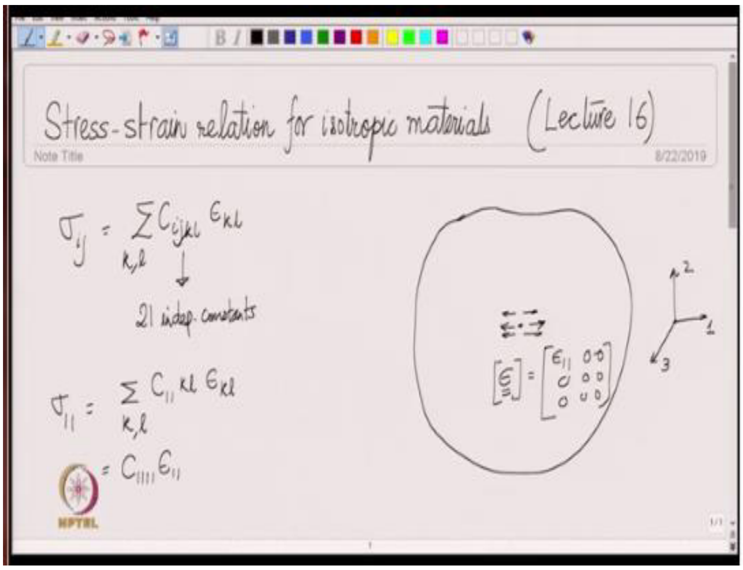

Figure 1: A body made up of an isotropic material with a coordinate system being stretched at an arbitrary point along $\underline{e}_1$ direction.

We have restricted the body in such a way that deformation is not allowed in any direction other than $\underline{e}_1$. Thus, the strain matrix at the point of stretching only has ϵ_{11} as a non-zero component, i.e.,

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

Using equation (1), we can write

$$\sigma_{11} = \sum_{k,l} C_{11kl}\ \epsilon_{kl} = C_{1111}\ \epsilon_{11} \tag{3}$$

We now do another experiment with the same body. However, this time instead of stretching along $\underline{e}_1$ direction, we stretch along $\underline{e}_2$ direction as shown in Figure 2. We again constrain the body to not allow deformation in any direction except $\underline{e}_2$.

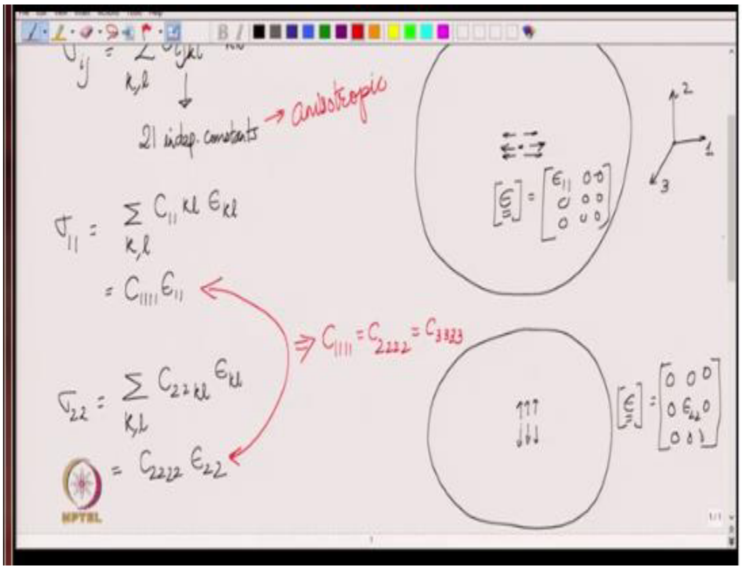

Figure 2: The same point in the same body as in Figure 1 is now stretched in the $\underline{e}_2$ direction.

Now, the new strain matrix at the point of stretching will be

$$\underline{\underline{\epsilon}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

Using equation (1), we can write:

$$\sigma_{22} = \sum_{k,l} C_{22kl}\ \epsilon_{kl} = C_{2222}\ \epsilon_{22} \tag{5}$$

As the material is isotropic, it has same property in all the directions. This also means that the stress generated in the two cases above should be the same if the strain applied is also the same. Thus, if ϵ_{11} in the first case equals ϵ_{22} in the second case, then σ_{11} in the first case should be same as σ_{22} in the second case. On comparing equations (3) and (5), we find that this is possible if and only if $C_{1111} = C_{2222}$. Similarly, if we do the analysis for stretching along $\underline{e}_3$, then we can conclude that

$$C_{1111} = C_{2222} = C_{3333} \tag{6}$$

These are additional constraints for the coefficients of stiffness tensor for isotropic materials. They are not covered by either minor or major symmetry. In fact, a rigorous analysis proves that there are several other constraints in this case all of which finally lead to only two independent constants for the stiffness tensor of isotropic materials. This also means that for isotropic materials, we just have to do two experiments to obtain its material constants and then generate the complete stress-strain relation (for a general material, we will accordingly need to do 21 experiments).

1.1 Stress-Strain relation (start time: 08:20)

Once we work out the stress-strain relation using a rigorous mathematical derivation, we get

$$\begin{aligned} \sigma_{ij} &= \lambda\, tr\left(\underline{\underline{\epsilon}}\right)\delta_{ij} + 2\mu\epsilon_{ij} \\ &= \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})\delta_{ij} + 2\mu\epsilon_{ij} \end{aligned} \tag{7}$$

Here (λ, μ) are called Lame's constants and are the two material constants for an isotropic material. Let us consider the component where $i = j = 1$, i.e.,

$$\sigma_{11} = \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2\mu\epsilon_{11} \tag{8}$$

Comparing with equation (1), we can see that C_{1111} (the constant relating σ_{11} and ϵ_{11}) will be given by the coefficient of ϵ_{11}, i.e.,

$$C_{1111} = \lambda + 2\mu \tag{9}$$

Likewise, we can also deduce C_{1122} and C_{1133} as

$$C_{1122} = C_{1133} = \lambda \tag{10}$$

Similarly, using equation (7), we can write

$$\sigma_{22} = \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2\mu\epsilon_{22} \Rightarrow C_{2222} = \lambda + 2\mu. \tag{11}$$

Comparing this with equation (9), we can verify that $C_{1111} = C_{2222}$ for isotropic materials. When $i \neq j$, the first term in equation (7) goes to zero because of the Kronecker delta function present in it. For example, σ_{12} will be

$$\sigma_{12} = 2\mu\epsilon_{12} = \mu(\epsilon_{12} + \epsilon_{21}) \Rightarrow C_{1212} = C_{1221} = \mu. \quad (12)$$

Equation (7) gives the relation where stress is expressed in terms of strain. Alternatively, we can use another form where strain is expressed in terms of stress. It is also called three-dimensional Hooke's law of Elasticity and can be written as

$$\epsilon_{11} = \frac{1}{E}\big(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\big) \quad (13)$$

$$\epsilon_{22} = \frac{1}{E}\big(\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})\big) \quad (14)$$

$$\epsilon_{33} = \frac{1}{E}\big(\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\big) \quad (15)$$

$$\gamma_{12} = 2\epsilon_{12} = \frac{\sigma_{12}}{G} = \frac{\tau_{12}}{G} \quad (16)$$

$$\gamma_{13} = \frac{\tau_{13}}{G} \quad (17)$$

$$\gamma_{23} = \frac{\tau_{23}}{G} \quad (18)$$

In this representation, we have three constants: *E* denoting Young's Modulus, *v* denoting Poisson's Ratio and *G* denoting shear modulus of elasticity. As we know that isotropic materials have only two independent constants. So, there is actually a relation between *E*, *G* and *v* given by

$$G = \frac{E}{2(1+\nu)} \quad (19)$$

Comparing equation (16) with equation (12), we can also immediately note that

$$\mu = G \quad (20)$$

1.2 Physical significance of *E*, *G* and *v* (start time: 15:54)

Consider equations (13), (14) and (15). We can see that the strain in one direction not only depends on the stress component in that direction but also on the stress components in other two directions. To visualize this, we can think of a simple experiment. Suppose that we have a rectangular beam of length *L*, breadth *B* and height *H* as shown in Figure 3. The beam is kept such that its length is along $\underline{e}_1$, its height is along $\underline{e}_2$ and i's breadth is along $\underline{e}_3$. We apply force on the left and right faces to stretch the beam.

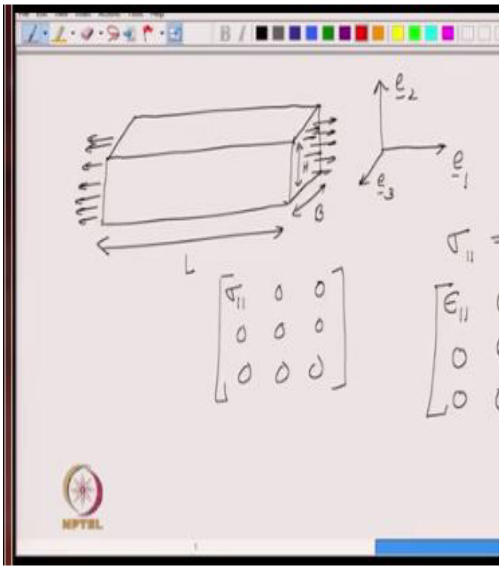

Figure 3: A rectangular beam is stretched by applying force through its left and right faces.

So, the stress component generated is σ_{11} (because force is in $\underline{e}_1$ direction on $\underline{e}_1$ plane). The shear components σ_{12} and σ_{13} will be zero. Also, as we are not applying any force on $\underline{e}_2$ and $\underline{e}_3$ planes, there is no stress on them. In fact, any section with normal along $\underline{e}_2$ and $\underline{e}_3$ will not have any traction component. The state of stress in this case will then be

$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{21}$$

This stress will lead to some strain in the body. We know that ϵ_{11} will be generated because the length of the beam will change. However, ϵ_{22} and ϵ_{33} also generates. Due to stretching in one direction, there will be contraction in the other two directions. We can realize this when we stretch a rubber bar or a soft bar along its axis. Their cross-section reduces when they are stretched. But, there will be no shear strains generated if we are careful in stretching in only one direction. Thus, the state of strain for the rectangular beam of Figure 3 will be

$$\begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \tag{22}$$

1.2.1 Young's Modulus (*E*) (start time: 22:10)

We know that local longitudinal strain (ϵ_{jj}) is given by $\frac{\partial u_j}{\partial X_j}$. If the elongation is uniform along the length of the bar, the local strain will be equal to the global strain. Thus:

$$\text{longitudinal strain} = \frac{\text{change in length}}{\text{length}} \tag{23}$$

Thus, we have:

$$\begin{aligned} \epsilon_{11} &= \frac{\Delta L}{L} \\ \epsilon_{22} &= \frac{\Delta H}{H} \\ \epsilon_{33} &= \frac{\Delta B}{B} \end{aligned} \tag{24}$$

If we now draw the graph of σ_{11} vs ϵ_{11} for the above experiment by measuring the change in length, we will get a curve as shown in Figure 4. The initial slope of this graph (shown as the red dotted line) gives us the Young's Modulus (E).

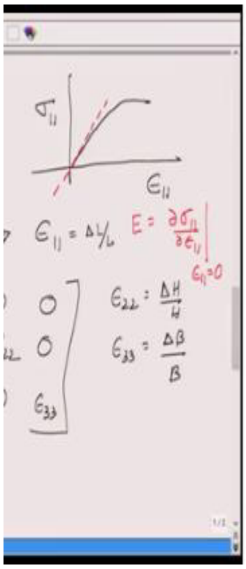

Figure 4: A typical plot of σ_{11} vs ϵ_{11} for the rectangular beam experiment of Figure 3

Thus, E is given by

$$E = \left.\frac{\partial \sigma_{11}}{\partial \epsilon_{11}}\right|_{\epsilon_{11}=0} \tag{25}$$

While computing this derivative, we should not have σ_{22} or σ_{33} present. Essentially, the beam should be stretched in such a way that it can freely shrink in the lateral directions. We can also prove the above formula using equation (13). We can put the conditions of the experiment into this, i.e. $\sigma_{22} = 0$ and $\sigma_{33} = 0$. So, we will get:

$$\sigma_{11} = E\epsilon_{11} \tag{26}$$

From this equation, we can see that the slope of the graph of σ_{11} vs ϵ_{11} will give us E. One thing to note here is that equation (13) gives us the slope of σ_{11} vs ϵ_{11} plot to be a constant. But in reality, the slope is not a constant as evident from Figure 4. This is because the linear stress strain relation (13) works only for small strains. On the other hand, Figure 4 shows the curve even for large strains and this is the reason that we had computed the slope of this graph at $\epsilon_{11} = 0$.

1.2.2 Poisson's Ratio (ν) (start time: 25:30)

The Poisson's ratio is defined as

$$\nu = -\frac{induced\ lateral\ normal\ strain}{imposed\ longitudinal\ strain} \tag{27}$$

In the rectangular beam experiment shown in Figure 3, we are directly imposing longitudinal strain in $\underline{e}_1$ direction and this, in turn, induces strain along $\underline{e}_2$ and $\underline{e}_3$ directions. For an isotropic body, the lateral strains in these directions will be equal. Thus, Poisson's ratio for this experiment will be

$$\nu = -\frac{\epsilon_{22}}{\epsilon_{11}} = -\frac{\epsilon_{33}}{\epsilon_{11}} \tag{28}$$

We can also derive this from the stress strain relations. Consider equations (13) and (14). We will substitute the conditions of the experiment, i.e. $\sigma_{22} = 0$ and $\sigma_{33} = 0$. From equation (13), we get

$$\epsilon_{11} = \frac{1}{E}\sigma_{11} \tag{29}$$

while from equation (14), we get

$$\begin{aligned} \epsilon_{22} &= \frac{1}{E}\big(0 - \nu(\sigma_{11} + 0)\big) \\ &= -\nu\frac{\sigma_{11}}{E} \\ &= -\nu\epsilon_{11} \quad \text{(using (29))} \end{aligned} \tag{30}$$

Using equations (29) and (30), we get to the same formula for Poisson's ratio as given in (28). So, to find the Poisson's ratio for a material, we need to stretch a bar made up of that material, measure the imposed and induced strains and then use equation (27).

1.2.3 Shear Modulus (G) (start time: 30:05)

From equation (16), we can see that shear modulus is given by

$$G = \frac{\tau_{12}}{\gamma_{12}} \tag{31}$$

So, if we can induce shear in a body and measure the corresponding shear stress, the ratio of stress to strain will give us the shear modulus. Let us conduct another experiment with the rectangular bar. The bar is clamped at the bottom face and we apply shear force on the top face (face with normal along $\underline{e}_2$) as shown in Figure 5. The force is along $\underline{e}_1$ direction.

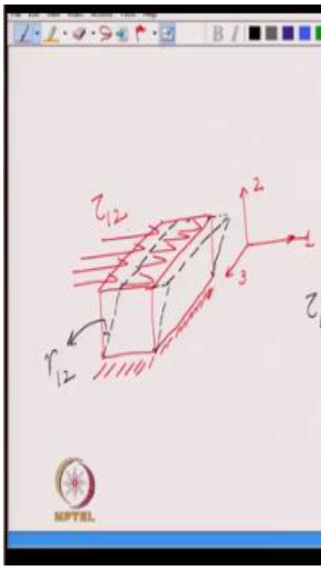

Figure 5: A rectangular bar is clamped at the bottom face. Shear force is applied at the top face. The coordinate system is also shown.

This effectively imposes shear stress τ_{12} in the body. Due to this, the bar shears (as shown in black in Figure 5). The initially perpendicular edges of the front face now get inclined and shear strain (γ_{12}) would be equal to the change in angle between these two edges (as shown in Figure 5). If we draw the plot for τ_{12} vs γ_{12}, we get a curve as shown in Figure 6.

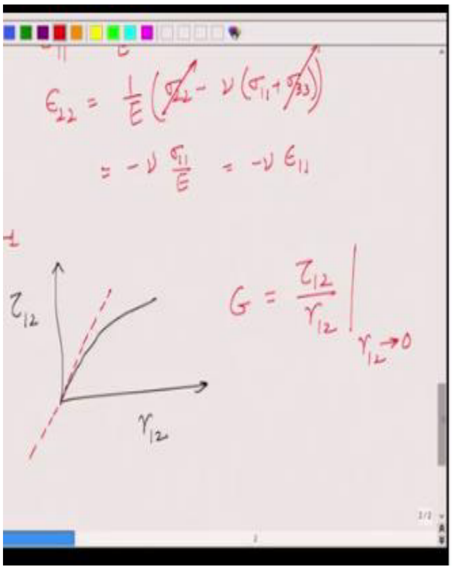

Figure 6: Plot of τ_{12} vs γ_{12} for the experiment shown in Figure 5

The initial slope of this curve will give us the shear modulus, i.e.

$$G = \left.\frac{\tau_{12}}{\gamma_{12}}\right|_{\gamma_{12}\to 0} \tag{32}$$

The initial slope is measured because the linear relations are valid only for small shear strains.

1.3 Bulk Modulus of Elasticity (K) (start time: 34:02)

We have discussed Young's modulus of elasticity and shear modulus of elasticity. Young's modulus relates normal strain with normal stress while shear modulus relates shear strain with shear stress. We will now discuss about bulk modulus of elasticity which relates volumetric strain with some kind of volumetric stress/pressure. Bulk modulus for gases is usually discussed in schools. When we apply pressure to any fluid (liquid/gas), the fluid volume decreases. This decrease can be quantified by volumetric strain given by

$$\text{Volumetric Strain} = \frac{\Delta V}{V} \tag{33}$$

If a pressure change of ΔP generates volumetric strain $\frac{\Delta V}{V}$ in a fiquid, bulk modulus is then given by

$$K_{liquid} = -\frac{\Delta P}{\Delta V/V} \tag{34}$$

For solids, $\frac{\Delta V}{V}$ is nothing but the volumetric strain discussed earlier, i.e.,

$$\frac{\Delta V}{V} = tr\left(\underline{\underline{\epsilon}}\right) = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \tag{35}$$

To obtain an equivalent of pressure in solids, we can use the decomposition of stress tensor, i.e.,

$$\underline{\underline{\sigma}} = \underbrace{\frac{1}{3} I_1 \underline{\underline{I}}}_{\text{hydrostatic part}} + \underbrace{\left(\underline{\underline{\sigma}} - \frac{1}{3} I_1 \underline{\underline{I}}\right)}_{\text{deviatoric part}} \tag{36}$$

The hydrostatic part is analogous to the pressure acting in liquids. As the stress tensor for liquids (in statics) is given by

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}}. \tag{37}$$

Comparing this with the hydrostatic part of stress for solids, we can conclude that the equivalent pressure p for solids is

$$p = -\frac{I_1}{3} \tag{38}$$

The negative sign comes because pressure is compressive in nature while the normal component of traction is tensile when positive. So, we can finally write:

$$K_{solids} = -\frac{-I_1/3}{tr\left(\underline{\underline{\epsilon}}\right)} = \frac{1}{3}\frac{tr\left(\underline{\underline{\sigma}}\right)}{tr\left(\underline{\underline{\epsilon}}\right)} \tag{39}$$

Let us use three-dimensional Hooke's law to obtain the formula for above quantity. Adding equations (13), (14) and (15), we get

$$\begin{aligned} \epsilon_{11} + \epsilon_{22} + \epsilon_{33} &= \frac{1}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) - \frac{2\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \\ \Rightarrow \quad tr\left(\underline{\underline{\epsilon}}\right) &= \frac{1-2\nu}{E} tr\left(\underline{\underline{\sigma}}\right) \\ \Rightarrow \quad K &= \frac{E}{3(1-2\nu)} \qquad \text{(using (39))} \end{aligned} \tag{40}$$

This is a very important relation. Firstly, it tells us that bulk modulus is not an independent constant. If we know the Young' modulus and the Poisson's ratio, we can get the Bulk modulus using the above relation. Secondly, this relation also gives an upper limit for the Poisson's ratio as discussed below.

1.4 Theoretical limits for the Poisson's Ratio (start time: 43:22)

The Poisson's ratio is usually positive as it is very difficult to find a material which when stretched in one direction, expands in the lateral directions also. We can observe that when ν is very close to $\frac{1}{2}$ in equation (40), the bulk modulus becomes very large which means that if we apply a finite amount of change in pressure, the volumetric strain that gets induced in the body is very small (also see (34)). This signifies incompressibility. Thus, $\nu \rightarrow \frac{1}{2}$ corresponds to the incompressible Limit. If this limit is crossed, K will become negative which is not physically meaningful. To obtain the lower limit for the Poisson's ratio, we can use equation (19), i.e.,

$$G = \frac{E}{2(1+\nu)} \tag{41}$$

From the rectangular beam experiment in Figure 3 and Figure 5, we can observe that when we apply a positive σ_{11}, ϵ_{11} should be positive and when we apply a positive τ_{12}, γ_{12} should be positive. So, Young's modulus and shear modulus are both positive quantities. As both the LHS and the numerator of the RHS in equation (41) are positive, the denominator of the RHS must also be positive, i.e.,

$$2(1+\nu) > 0 \Rightarrow 1+\nu > 0 \Rightarrow \nu > -1 \tag{42}$$

We thus have the following theoretical limits for the Poisson's ratio, i.e.,

$$\boxed{-1 < \nu \leq \frac{1}{2}} \tag{43}$$

which holds only for isotropic materials.

2 Other types of materials (start time: 50:55)

There are materials which are not isotropic but commonly seen in nature. Let us consider an isotropic material as shown in Figure 7. Fibers are engraved in the material to make it a fiber re-inforced material.

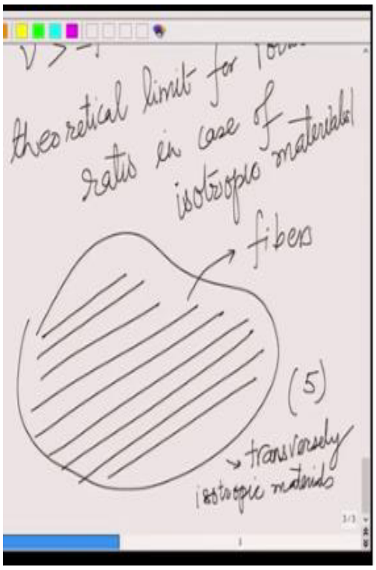

Figure 7: An initially isotropic material is engraved with fibers

When we put fibers in the material, the Young's modulus of the material in the direction of the fiber would become different to the Young's modulus in the direction perpendicular to the fibers. Actually, in all directions transverse to the fibers, it has same young's modulus but different from the one along the fiber. Due to this directional dependence, the material no longer remains isotropic. Such materials are called transversely isotropic and they have 5 independent material constants. Let us now put another family of fibers in the material but perpendicular to the initial family as shown in Figure 8.

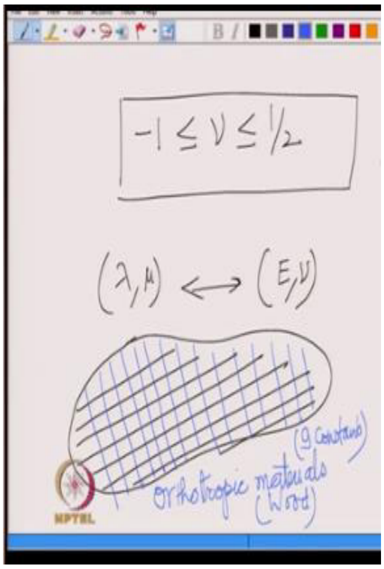

Figure 8: An initially isotropic material is engraved with two families of fibers perpendicular to each other

Assume the two fibers have different properties. So, the Young's modulus along the fibers will be different. Also, the Young's modulus in the direction perpendicular to both the fibers will be different. Such materials are called orthotropic materials and they have 9 independent constants. Wood is an example of an orthotropic material. We should note that all the discussion in this lecture is valid only for elastic materials where stress depends only on the current value of strain. There are other materials where stress not only depends on strain but also on strain gradients. We can also have elastoplastic materials where we have plastic strain in addition to elastic strain. In this course however, we will mostly discuss isotropic elastic materials.