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Chapter 9

ENERGY METHODS; Part I

9.1 Introduction

1 Energy methods are powerful techniques for both formulation (of the stiffness matrix of an element[1]) and for the analysis (i.e. deflection) of structural problems.

2 We shall explore two techniques:

1. Real Work
2. Virtual Work (Virtual force)

9.2 Real Work

3 We start by revisiting the first law of thermodynamics:

> The time-rate of change of the total energy (i.e., sum of the kinetic energy and the internal energy) is equal to the sum of the rate of work done by the external forces and the change of heat content per unit time.

$$\boxed{\frac{d}{dt}(K+U) = W_e + H} \tag{9.1}$$

where K is the kinetic energy, U the internal strain energy, W_e the external work, and H the heat input to the system.

4 For an adiabatic system (no heat exchange) and if loads are applied in a quasi static manner (no kinetic energy), the above relation simplifies to:

$$\boxed{W_e = U} \tag{9.2}$$

5 Simply stated, the first law stipulates that the external work must be equal to the internal strain energy due to the external load.

[1]More about this in *Matrix Structural Analysis*.

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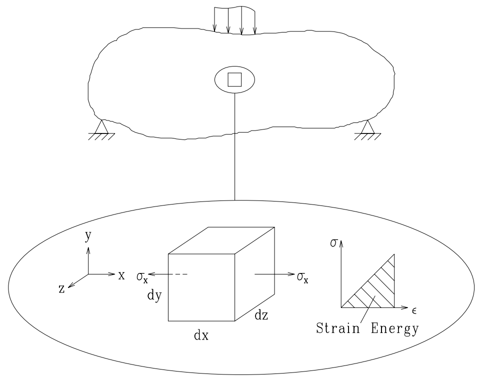

Figure 9.2: Strain Energy Definition

9.2.2 Internal Work

9 Considering an infinitesimal element from an arbitrary structure subjected to uniaxial state of stress, the strain energy can be determined with reference to Fig. 9.2. The net force acting on the element while deformation is taking place is $P = \sigma_x dydz$. The element will undergo a displacement $u = \varepsilon_x dx$. Thus, for a linear elastic system, the strain energy density is $dU = \frac{1}{2}\sigma\varepsilon$. And the total strain energy will thus be

$$\boxed{U = \frac{1}{2}\int_{\text{Vol}} \varepsilon \underbrace{E\varepsilon}_{\sigma} d\text{Vol}} \tag{9.7}$$

10 When this relation is applied to various structural members it would yield:

Axial Members:

$$\left.\begin{aligned} U &= \int_{\text{Vol}} \frac{\varepsilon\sigma}{2} d\text{Vol} \\ \sigma &= \frac{P}{A} \\ \varepsilon &= \frac{P}{AE} \\ dV &= Adx \end{aligned}\right\} \boxed{U = \int_0^L \frac{P^2}{2AE}dx} \tag{9.8}$$

Torsional Members:

$$\left.\begin{aligned} U &= \tfrac{1}{2}\int_{\text{Vol}} \varepsilon \underbrace{E\varepsilon}_{\sigma} d\text{Vol} \\ U &= \tfrac{1}{2}\int_{\text{Vol}} \gamma_{xy} \underbrace{G\gamma_{xy}}_{\tau_{xy}} d\text{Vol} \\ \tau_{xy} &= \tfrac{Tr}{J} \\ \gamma_{xy} &= \tfrac{\tau_{xy}}{G} \\ d\text{Vol} &= rd\theta drdx \\ J &= \int_o^r \int_0^{2\pi} r^2 d\theta\, dr \end{aligned}\right\} \boxed{U = \int_0^L \frac{T^2}{2GJ}dx} \tag{9.9}$$

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9.3 Virtual Work

11 A severe limitation of the method of real work is that only deflection along the externally applied load can be determined.

12 A more powerful method is the virtual work method.

13 The principle of Virtual Force (VF) relates *force* systems which satisfy the requirements of *equilibrium*, and *deformation* systems which satisfy the requirement of *compatibility*

14 In any application the force system could either be the actual set of **external** loads $d\mathbf{p}$ or some **virtual** force system which happens to satisfy the condition of *equilibrium* $\delta\overline{\mathbf{p}}$. This set of external forces will induce internal actual forces $d\boldsymbol{\sigma}$ or internal virtual forces $\delta\overline{\boldsymbol{\sigma}}$ compatible with the externally applied load.

15 Similarly the deformation could consist of either the actual joint deflections $d\mathbf{u}$ and compatible internal deformations $d\boldsymbol{\varepsilon}$ of the structure, or some **virtual** external and internal deformation $\delta\overline{\mathbf{u}}$ and $\delta\overline{\boldsymbol{\varepsilon}}$ which satisfy the conditions of *compatibility*.

16 It is often simplest to assume that the **virtual load is a unit load**.

17 Thus we may have 4 possible combinations, Table 9.1: where: d corresponds to the actual,

	Force		Deformation		IVW	Formulation
	External	Internal	External	Internal		
1	$d\mathbf{p}$	$d\boldsymbol{\sigma}$	$d\mathbf{u}$	$d\boldsymbol{\varepsilon}$		
2	$\delta\overline{\mathbf{p}}$	$\delta\overline{\boldsymbol{\sigma}}$	$d\mathbf{u}$	$d\boldsymbol{\varepsilon}$	$\delta\overline{U}^*$	Flexibility
3	$d\mathbf{p}$	$d\boldsymbol{\sigma}$	$\delta\overline{\mathbf{u}}$	$\delta\overline{\boldsymbol{\varepsilon}}$	$\delta\overline{U}$	Stiffness
4	$\delta\overline{\mathbf{p}}$	$\delta\overline{\boldsymbol{\sigma}}$	$\delta\overline{\mathbf{u}}$	$\delta\overline{\boldsymbol{\varepsilon}}$		

Table 9.1: Possible Combinations of Real and Hypothetical Formulations

and δ (with an overbar) to the hypothetical values.

This table calls for the following observations

1. The second approach is the same one on which the method of virtual or unit load is based. It is simpler to use than the third as a internal force distribution compatible with the assumed virtual force can be easily obtained for statically determinate structures. This approach will yield exact solutions for statically determinate structures.

2. The third approach is favored for statically indeterminate problems or in conjunction with approximate solution. It requires a proper "guess" of a displacement shape and is the basis of the stiffness method.

18 Let us consider an arbitrary structure and load it with both real and virtual loads in the following sequence, Fig. 9.4. For the sake of simplicity, let us assume (or consider) that this structure develops only axial stresses.

1. If we apply the virtual load, then

$$\frac{1}{2}\delta\overline{P}\delta\overline{\Delta} = \frac{1}{2}\int_{d\mathrm{Vol}} \delta\overline{\sigma}\delta\overline{\varepsilon}d\mathrm{Vol} \tag{9.12}$$