Chapter 10: Fourier Cosine and Sine Transforms

Introduction

In civil engineering, analyzing and solving boundary value problems—especially those involving heat transfer, wave motion, or vibrations—often requires converting a function from the spatial domain to a frequency domain. Fourier transforms are indispensable in this context. However, when the problem is defined on a semi-infinite domain (e.g., $x \geq 0$), **Fourier Cosine and Sine Transforms** are preferred over the general Fourier transform. These transforms allow decomposition of functions into orthogonal sine and cosine basis functions, effectively handling problems with specific boundary conditions.

10.1 Fourier Cosine Transform (FCT)

Definition

For a function $f(x)$ defined on $[0, \infty)$, the **Fourier Cosine Transform** is defined as:

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(sx)\, dx$$

Where:

- $f(x)$ is piecewise continuous on every finite interval in $[0, \infty)$,
- $f(x)$ is absolutely integrable on $[0, \infty)$,
- s is the transform variable (frequency domain variable).

Inverse Fourier Cosine Transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos(sx)\, ds$$

This restores the original function from its cosine transform.

Properties of Fourier Cosine Transform

1. **Linearity**:

$$\mathcal{F}_c\{af(x) + bg(x)\} = a\mathcal{F}_c\{f(x)\} + b\mathcal{F}_c\{g(x)\}$$

2. **Scaling**:

$$\mathcal{F}_c\{f(ax)\} = \frac{1}{a} F_c\left(\frac{s}{a}\right), \quad a > 0$$

3. **Differentiation**: If $f(x)$ is differentiable and vanishes as $x \to \infty$,

$$\mathcal{F}_c\{f'(x)\} = -s\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin(sx)\,dx$$

4. **Parseval's Identity**:

$$\int_0^\infty f(x)^2\,dx = \int_0^\infty F_c(s)^2\,ds$$

Examples

Example 1: Fourier Cosine Transform of $f(x) = e^{-ax}$, where $a > 0$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\cos(sx)\,dx = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2 + s^2}$$

10.2 Fourier Sine Transform (FST)

Definition

For a function $f(x)$ defined on $[0, \infty)$, the **Fourier Sine Transform** is defined as:

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin(sx)\,dx$$

Inverse Fourier Sine Transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s)\sin(sx)\,ds$$

Properties of Fourier Sine Transform

1. **Linearity**:

$$\mathcal{F}_s\{af(x) + bg(x)\} = a\mathcal{F}_s\{f(x)\} + b\mathcal{F}_s\{g(x)\}$$

2. **Scaling**:

$$\mathcal{F}_s\{f(ax)\} = \frac{1}{a} F_s\left(\frac{s}{a}\right), \quad a > 0$$

3. **Differentiation**: If $f(x)$ is differentiable and vanishes as $x \to \infty$,

$$\mathcal{F}_s\{f'(x)\} = s\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(sx)\,dx$$

4. **Parseval's Identity**:

$$\int_0^\infty f(x)^2\,dx = \int_0^\infty F_s(s)^2\,ds$$

Examples

Example 2: Fourier Sine Transform of $f(x) = e^{-ax}$, where $a > 0$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\sin(sx)\,dx = \sqrt{\frac{2}{\pi}} \cdot \frac{s}{a^2 + s^2}$$

10.3 Applications in Civil Engineering

Fourier sine and cosine transforms are applied in the following civil engineering contexts:

- **Heat Conduction in Semi-Infinite Slabs**: Boundary conditions at one end (e.g., insulated or fixed temperature) often lead to cosine or sine transform solutions.
- **Deflection of Beams with One Fixed End**: Solving beam bending equations using Fourier cosine transforms when the displacement or slope is known at one end.
- **Wave Propagation in Strings or Rods**: For rods fixed at one end and free at the other, sine transforms help solve the partial differential equations.

10.4 Relation to Full Fourier Transform

The full Fourier transform is defined over $(-\infty, \infty)$, but for even and odd extensions of functions defined only on $[0, \infty)$, the sine and cosine transforms correspond respectively:

- If $f(x)$ is even:

$$\mathcal{F}\{f(x)\} \leftrightarrow \text{Cosine Transform}$$

- If $f(x)$ is odd:

$$\mathcal{F}\{f(x)\} \leftrightarrow \text{Sine Transform}$$

Thus, these transforms are not just computational tricks—they carry deep connections to symmetry properties and domain constraints of physical systems.

10.5 Standard Fourier Cosine and Sine Transform Pairs

Function $f(x)$	Fourier Cosine Transform $F_c(s)$	Fourier Sine Transform $F_s(s)$
1	$\sqrt{\frac{2}{\pi}} \cdot \frac{1}{s}$	$\sqrt{\frac{2}{\pi}} \cdot \frac{1}{s}$
e^{-ax}	$\sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2}$	$\sqrt{\frac{2}{\pi}} \cdot \frac{s}{a^2+s^2}$
x	$\sqrt{\frac{2}{\pi}} \cdot \frac{s}{s^2}$	$\sqrt{\frac{2}{\pi}} \cdot \frac{2a}{(a^2+s^2)^2}$
$\sin(ax)$	$\sqrt{\frac{2}{\pi}} \cdot \frac{a}{s^2+a^2}$	$\sqrt{\frac{2}{\pi}} \cdot \frac{s}{s^2+a^2}$

10.6 Advanced Applications in Boundary Value Problems

Many engineering problems reduce to solving **partial differential equations (PDEs)** with boundary conditions. Fourier sine and cosine transforms are highly effective in solving such problems, particularly when domains are semi-infinite.

10.6.1 Application: Heat Equation in a Semi-Infinite Rod

Consider a semi-infinite rod $x \in [0, \infty)$ initially at zero temperature, and for $t > 0$, the end $x = 0$ is held at a constant temperature T_0. The governing equation is the **heat conduction equation**:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad x > 0,\ t > 0$$

Subject to:

$$u(0,t) = T_0, \quad u(x,0) = 0, \quad \lim_{x\to\infty} u(x,t) = 0$$

Solution Method:

We take the **Fourier Cosine Transform** with respect to x:

Let $U(s,t) = \mathcal{F}_c\{u(x,t)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t)\cos(sx)\,dx$

Using properties of derivatives under transforms:

$$\mathcal{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -s^2 U(s,t)$$

$$\Rightarrow \frac{\partial U}{\partial t} = -\alpha^2 s^2 U(s,t)$$

This is a first-order linear ODE in t with solution:

$$U(s,t) = A(s)e^{-\alpha^2 s^2 t}$$

From initial condition $u(x,0) = 0 \Rightarrow U(s,0) = 0 \Rightarrow A(s) = 0$, but this contradicts the boundary condition. To fix this, we transform the nonhomogeneous boundary via suitable substitution (e.g., Duhamel's principle or separation of variables), which leads to:

$$u(x,t) = T_0 \,\mathrm{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right)$$

Where $\mathrm{erfc}(z)$ is the complementary error function.

10.6.2 Application: Beam Deflection with One Fixed End

A cantilever beam of length L, fixed at $x = 0$, subjected to a load $q(x)$. The **Euler-Bernoulli beam equation** is:

$$\frac{d^4 y}{dx^4} = \frac{q(x)}{EI}$$

With boundary conditions (fixed end at $x = 0$):

$$y(0) = 0, \quad y'(0) = 0$$

We apply the **Fourier Cosine Transform** to both sides:

Let $Y(s) = \mathcal{F}_c\{y(x)\}$, then:

$$\mathcal{F}_c\left\{\frac{d^4y}{dx^4}\right\} = s^4Y(s)$$

Thus,

$$s^4Y(s) = \frac{1}{EI}\mathcal{F}_c\{q(x)\} \quad \Rightarrow \quad Y(s) = \frac{1}{EIs^4}\mathcal{F}_c\{q(x)\}$$

Taking the inverse transform yields the deflection $y(x)$.

10.7 Solving PDEs Using Fourier Sine Transform

Now consider problems where the solution vanishes at the boundary $x = 0$, which is ideal for **Fourier Sine Transform**.

10.7.1 Wave Equation with a Free End

Let us consider the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

With boundary conditions:

$$u(0,t) = 0, \quad \lim_{x\to\infty} u(x,t) = 0, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = 0$$

Apply the **Fourier Sine Transform**:

$$U(s,t) = \mathcal{F}_s\{u(x,t)\} \Rightarrow \frac{\partial^2 U}{\partial t^2} = -c^2s^2U$$

This is a second-order ODE in t:

$$\Rightarrow U(s,t) = A(s)\cos(cst) + B(s)\sin(cst)$$

Initial conditions give:

$$U(s,0) = \mathcal{F}_s\{f(x)\} = A(s), \qquad \left.\frac{\partial U}{\partial t}\right|_{t=0} = 0 \Rightarrow B(s) = 0$$

Thus,

$$U(s,t) = \mathcal{F}_s\{f(x)\} \cdot \cos(cst) \Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \mathcal{F}_s\{f(x)\} \cos(cst) \sin(sx) \, ds$$

This integral solution gives the **displacement of a vibrating rod** fixed at one end.

10.8 Evaluation of Integrals Using Transforms

Another practical use of Fourier sine and cosine transforms is evaluating **improper integrals**.

Example 3: Evaluate $\int_0^\infty \frac{x \sin(ax)}{x^2+b^2} dx$

Let:

$$f(x) = \frac{x}{x^2+b^2} \Rightarrow \text{FST of } f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x \sin(sx)}{x^2+b^2} \, dx$$

Using integral tables or residue calculus:

$$\int_0^\infty \frac{x \sin(ax)}{x^2+b^2} dx = \frac{\pi}{2} e^{-ab}$$

Hence, Fourier transforms provide a powerful way to evaluate such definite integrals analytically.

10.9 Fourier Transforms of Derivatives

Knowing how to handle **derivatives** within the transforms is crucial for PDEs.

10.9.1 Cosine Transform of First Derivative

$$\mathcal{F}_c\{f'(x)\} = -s\mathcal{F}_s\{f(x)\}$$

10.9.2 Sine Transform of First Derivative

$$\mathcal{F}_s\{f'(x)\} = s\mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}f(0)$$

These relations help simplify many boundary-value problems in civil engineering (e.g., thermal gradient, slope in beam deflection).
