Chapter 5: Degrees of Freedom and SDOF Idealization

Introduction

In the context of earthquake engineering, the concept of *degrees of freedom* (DOF) plays a foundational role in structural modeling and analysis. The accurate representation of structural behavior under seismic forces begins with understanding how a structure can move—this is captured through its degrees of freedom. For many analysis methods, particularly in dynamic analysis, simplifying complex structures into *single-degree-of-freedom (SDOF)* systems helps in understanding fundamental behavior before extending to more realistic models. This chapter delves into the conceptual framework and mathematical formulation associated with DOF and SDOF idealization in seismic analysis.

5.1 Degrees of Freedom (DOF)

5.1.1 Definition

A degree of freedom refers to the minimum number of independent coordinates required to define the motion of a system. In structural engineering, this typically relates to possible displacements (translational or rotational) that a structure can experience.

5.1.2 Types of Degrees of Freedom

- **Translational DOF:** Movement along x, y, or z directions.
- **Rotational DOF:** Rotation about x, y, or z axes.
- **Coupled DOFs:** Some structural systems exhibit coupling between translation and rotation, especially in irregular or torsionally unbalanced buildings.

5.1.3 Importance in Earthquake Engineering

- Determines the complexity of structural analysis.
- Influences the natural frequencies and mode shapes.

- Helps in selecting appropriate numerical methods (modal analysis, time history, etc.).

5.2 Dynamic Degrees of Freedom

5.2.1 Definition

Dynamic degrees of freedom are those coordinates which define the motion of a structure due to dynamic (time-varying) loads such as seismic ground motion.

5.2.2 Determination

To identify the number of dynamic DOFs:

- Examine the geometry of the structure.
- Identify possible independent displacement points.
- Apply constraints and supports.

5.2.3 Examples

- **Cantilever Beam:** A vertical cantilever with a mass at the top may have one dynamic DOF—lateral displacement at the top.
- **Multi-story Frames:** Each floor may have lateral DOFs, resulting in a multi-degree-of-freedom system.

5.3 Lumped Mass Idealization

5.3.1 Concept

In real structures, mass is distributed throughout. For simplicity in dynamic analysis:

- Mass is assumed to be *lumped* at specific points (commonly at floor levels).
- Floors are assumed to be rigid in their own plane.

5.3.2 Justification

- Acceptable in buildings where floor stiffness is high.
- Allows reduction of complex systems to simpler models.

5.3.3 Application

Lumped mass models are extensively used in:

- Modal analysis
- Time history analysis
- Response spectrum analysis

5.4 SDOF System: Formulation and Idealization

5.4.1 What is an SDOF System?

A *Single Degree of Freedom (SDOF)* system is the simplest dynamic model where the motion of the system can be described using a single coordinate, typically lateral displacement.

5.4.2 SDOF Elements

- **Mass (m):** Represents inertia.
- **Stiffness (k):** Represents restoring force.
- **Damping (c):** Represents energy dissipation (optional in ideal models).
- **Displacement (u):** Time-dependent motion variable.

5.4.3 Equation of Motion for Undamped SDOF

$$m\ddot{u}(t)+ku(t)=-m\ddot{u}_g(t)$$

Where:

- $\ddot{u}(t)$ = absolute acceleration of the mass,
- $\ddot{u}_g(t)$ = ground acceleration.

5.4.4 With Damping

$$m\ddot{u}(t)+c\dot{u}(t)+ku(t)=-m\ddot{u}_g(t)$$

5.4.5 Assumptions in SDOF Idealization

- Building floors are infinitely rigid in their own plane.
- Masses are lumped at floor levels.
- Only lateral displacements are considered (for seismic loads).
- Linear elastic behavior unless specified otherwise.

5.5 Idealization of Structures as SDOF Systems

5.5.1 When Can Structures Be Idealized as SDOF?

- Regular low-rise buildings.
- Structures dominated by one vibration mode.
- Structures with one mass-concentrated location.

5.5.2 Steps for Idealization

1. Identify the primary direction of motion.
2. Lump the mass at a specific level (e.g., roof level).
3. Determine the equivalent stiffness of the structure.
4. Apply seismic load as base acceleration.

5.5.3 Effective Parameters

- **Effective Mass m_e:** Mass participating in the mode.
- **Effective Stiffness k_e:** Equivalent lateral stiffness.
- **Effective Damping c_e:** Damping ratio based on energy dissipation.

5.6 Damped and Undamped Systems

5.6.1 Undamped Free Vibration

$$\ddot{u}(t)+\omega_n^2 u(t)=0$$

Solution:

$$u(t)=A\cos(\omega_n t)+B\sin(\omega_n t)$$

Where $\omega_n=\sqrt{\frac{k}{m}}$

5.6.2 Damped Free Vibration

$$\ddot{u}(t)+2\zeta\omega_n\dot{u}(t)+\omega_n^2 u(t)=0$$

- ζ: damping ratio
- Behavior depends on value of ζ:
  - Underdamped ($\zeta<1$)
  - Critically damped ($\zeta=1$)

- Overdamped ($\zeta > 1$)

5.7 Response of SDOF Systems to Ground Motion

5.7.1 Seismic Excitation

The base of the structure moves due to ground motion. The equation becomes:

$$m\ddot{u}(t)+c\dot{u}(t)+ku(t)=-m\ddot{u}_g(t)$$

5.7.2 Relative Displacement

$$u_r(t)=u(t)-u_g(t)$$

Where:

- u_r = displacement of mass relative to ground
- The response depends on natural period and damping.

5.7.3 Numerical Solution Techniques

- Time-stepping methods (Newmark-beta, Runge-Kutta)
- Frequency domain solutions (Fourier transform)
- Software tools (e.g., MATLAB, SAP2000)

5.8 Significance of SDOF Models in Seismic Design

5.8.1 Understanding Fundamental Response

- Allows visualization of how buildings respond to seismic forces.
- Simplifies understanding of resonance, damping effects, and base isolation.

5.8.2 Base for Design Spectra

- Design response spectra are derived using SDOF models.
- Helps in quick estimation of seismic demand on structures.

5.8.3 Educational Value

- Serves as a starting point before MDOF and nonlinear analysis.
- Essential in academic curriculum and foundational research.

5.9 Limitations of SDOF Idealization

5.9.1 Oversimplification of Structural Behavior

- SDOF models assume the entire structure vibrates in a single mode.
- In reality, multi-story or irregular structures exhibit **multi-modal** responses.
- Higher modes can significantly influence shear forces, overturning moments, and inter-story drift in tall or flexible buildings.

5.9.2 Neglect of Torsional Effects

- SDOF idealization often ignores torsional behavior.
- In asymmetric structures or those with eccentric mass/stiffness distributions, torsional effects can lead to critical failure mechanisms during seismic events.

5.9.3 Inability to Capture Localized Deformations

- SDOF models represent global lateral displacement only.
- Cannot reflect **local failures**, **beam-column interactions**, or **floor diaphragm flexibility**.

5.10 Comparison between SDOF and MDOF Systems

Aspect	SDOF System	MDOF System
Degrees of Freedom	1	>1 (typically one per floor)
Analysis Complexity	Simple	Computationally intensive
Modal Participation	Dominated by first mode	Includes multiple modes
Suitable for	Regular, low-rise buildings	Tall, irregular, or complex structures
Torsional Effects	Not considered	Can be modeled explicitly
Accuracy	Approximate	More accurate and realistic

5.11 Idealization of Real Structures as SDOF

5.11.1 Equivalence through Modal Analysis

- From a **modal analysis**, a structure's response can be broken into modes.
- If the **first mode** contributes >90% of the mass participation, the structure can be idealized as an **SDOF system** for practical purposes.

5.11.2 Fundamental Period Estimation

For SDOF idealization, **natural period** T is critical:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

In real structures:

- Use **empirical formulas** (e.g., IS 1893:2016) to estimate T
- Convert structure to an equivalent SDOF system by using **effective stiffness and mass**

5.11.3 Participation Factor (Γ)

Defines how much of the total mass participates in a specific mode:

$$\Gamma = \frac{\Phi^T M 1}{\Phi^T M \Phi}$$

Where:

- Φ = mode shape vector
- M = mass matrix
- 1 = influence vector (usually all 1s for lateral loads)

5.12 Concept of Modal Mass and Modal Stiffness

5.12.1 Modal Mass $m^{¿}$

$$m^{¿} = \Phi^T M \Phi$$

This is the mass associated with a given mode. It helps in translating MDOF systems to an equivalent SDOF system.

5.12.2 Modal Stiffness $k^{¿}$

$$k^{¿} = m^{¿} \omega^2$$

Where ω is the circular frequency of that mode.

5.13 Response Spectrum Analysis using SDOF Systems

5.13.1 Basic Principle

Response spectra are derived by subjecting an SDOF system (with varying periods and damping) to a specific ground motion and plotting peak responses.

5.13.2 Use in Codes

Design response spectra in building codes (like IS 1893) are **based on SDOF behavior** under standardized seismic input.

5.13.3 Pseudo vs Actual Spectra

- **Pseudo Spectral Acceleration (PSA)**: $PSA = \omega^2 \cdot u_{max}$
- **Spectral Displacement (SD)**: Peak relative displacement
- **Spectral Velocity (SV)**: Peak relative velocity

5.14 Use of SDOF Systems in Seismic Isolation and Energy Dissipation

5.14.1 Seismic Isolation Modeling

- Base-isolated structures are often modeled as **two-mass SDOF systems**: superstructure + isolator.
- Effective period of the system increases → reduces acceleration response.

5.14.2 Dampers and Energy Dissipating Devices

- SDOF models are used to evaluate **energy dissipation capacity**.
- Devices like viscous dampers, yielding braces, or tuned mass dampers are first tested on SDOF systems before integration into MDOF frameworks.

5.15 SDOF Systems in Time History Analysis

5.15.1 Time-Stepping Solution

Numerical methods used for solving the SDOF response to ground motion:

- **Newmark-beta method**
- **Wilson-θ method**
- **Runge-Kutta method**

5.15.2 Nonlinear SDOF Models

- For realistic seismic analysis, SDOF systems are modeled with **bilinear or elasto-plastic behavior**.
- Hysteresis loops (force–displacement) show **energy dissipation**, **ductility**, and **residual deformation**.

5.16 Advanced Applications of SDOF Idealization

5.16.1 Performance-Based Design

- Each performance level (Immediate Occupancy, Life Safety, Collapse Prevention) is linked to **SDOF demand** using capacity spectrum method.

5.16.2 Displacement-Based Seismic Design

- Focuses on displacement rather than force.
- Uses **equivalent SDOF systems** to match demand and capacity spectra.

5.16.3 Pushover Analysis

- The non-linear static analysis method.
- Load applied incrementally until target displacement (SDOF-based).
- Converts actual structure to a capacity curve of an equivalent SDOF system.
