Solid Mechanics
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Lecture - 19
Strain Matrix in Cylindrical Coordinate System

Hello everyone! Welcome to Lecture 19! In this lecture, we will see how the strain tensor is represented as a matrix in cylindrical coordinate system. In the previous two lectures, we had derived the equations of equilibrium in cylindrical coordinate system and the idea was to solve the deformation problem in cylindrical coordinate system. Later on, we will also relate stress with strain in this coordinate system.

1 Strain matrix in Cylindrical Coordinate System (start time: 01:02)

The strain tensor ($\underline{\underline{\epsilon}}$) is defined as

$$\underline{\underline{\epsilon}} = \frac{1}{2}\left(\underline{\nabla}\,\underline{u} + \underline{\nabla}\,\underline{u}^T\right) \tag{1}$$

We can recall its matrix form in Cartesian coordinate system:

$$\left[\underline{\underline{\epsilon}}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} \dfrac{\partial u_1}{\partial X_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_2} + \dfrac{\partial u_2}{\partial X_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_3} + \dfrac{\partial u_3}{\partial X_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_2} + \dfrac{\partial u_2}{\partial X_1}\right) & \dfrac{\partial u_2}{\partial X_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_3} + \dfrac{\partial u_3}{\partial X_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_3} + \dfrac{\partial u_3}{\partial X_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_3} + \dfrac{\partial u_3}{\partial X_2}\right) & \dfrac{\partial u_3}{\partial X_3} \end{bmatrix} \tag{2}$$

Let us work out the same in cylindrical coordinate system.

1.1 Representation of gradient (start time: 03:03)

We need to first express gradients in cylindrical coordinate system. In Cartesian coordinate system, the gradient of a quantity is given by

$$\underline{\nabla}(\cdot) = \frac{\partial}{\partial X_1}(\cdot)\otimes\underline{e}_1 + \frac{\partial}{\partial X_2}(\cdot)\otimes\underline{e}_2 + \frac{\partial}{\partial X_3}(\cdot)\otimes\underline{e}_3 \tag{3}$$

whereas its definition in cylindrical coordinate system is

$$\underline{\nabla}(\cdot) = \frac{\partial}{\partial r}(\cdot) \otimes \underline{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}(\cdot) \otimes \underline{e}_\theta + \frac{\partial}{\partial z}(\cdot) \otimes \underline{e}_z \tag{4}$$

Notice that the partial derivative with respect to θ is divided by r as θ is non-dimensional and we are taking the gradient in space. We will use the above form to obtain the gradient of the displacement vector $\underline{u}$.

1.2 Representation of displacement (start time: 05:05)

The displacement vector can be written in cylindrical coordinate system by decomposing it along cylindrical basis as follows:

$$\underline{u} = u_r \underline{e}_r + u_\theta \underline{e}_\theta + u_z \underline{e}_z \tag{5}$$

To understand the physical significance of various components, a section of an arbitrary body (in its reference configuration) parallel to z plane is shown in Figure 1.

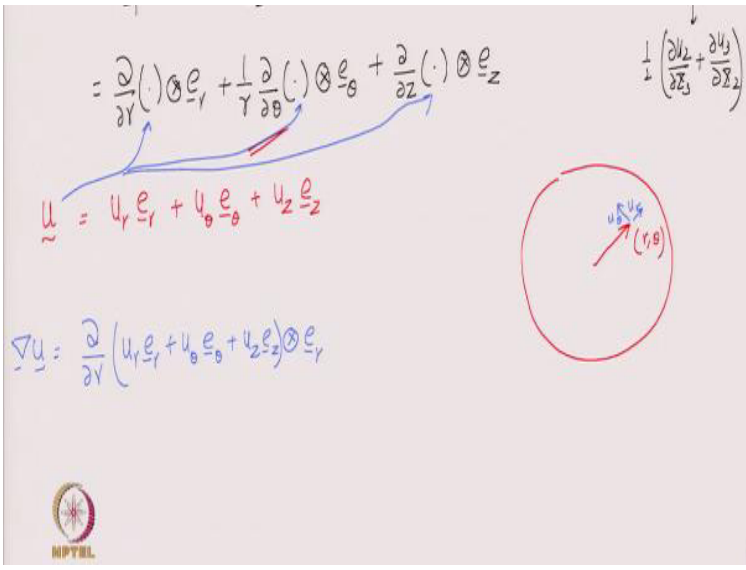

Figure 1: A typical section of an arbitrary body parallel to *z* plane: a point with coordinates (r,θ) with respect to the origin of cylindrical coordnate system is also shown along with the components of displacement along basis directions.

A point with coordinates (r,θ) is also shown. After deformation of the body, the displacement of this point in the radial direction is u_r, displacement in the θ direction is u_θ and displacement in the z direction (coming out of the plane) is u_z.

1.3 Representation of displacement gradient (start time: 06:27)

Now, we can plug in the displacement vector given in equation (5) in the gradient definition (4) to obtain the following:

$$\begin{aligned}\underline{\nabla}\ \underline{u} &= \frac{\partial}{\partial r}(u_r\underline{e}_r + u_\theta\underline{e}_\theta + u_z\underline{e}_z)\otimes\underline{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}(u_r\underline{e}_r + u_\theta\underline{e}_\theta + u_z\underline{e}_z)\otimes\underline{e}_\theta \\ &+ \frac{\partial}{\partial z}(u_r\underline{e}_r + u_\theta\underline{e}_\theta + u_z\underline{e}_z)\otimes\underline{e}_z.\end{aligned} \tag{6}$$

For the $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial z}$ terms here, the basis vectors act as a constant as they change only with θ. The derivatives of basis vectors with respect to θ were derived earlier which are

$$\frac{\partial\underline{e}_r}{\partial\theta} = \underline{e}_\theta, \qquad \frac{\partial\underline{e}_\theta}{\partial\theta} = -\underline{e}_r, \qquad \frac{\partial\underline{e}_z}{\partial\theta} = \underline{0}. \tag{7}$$

Upon substituting them in equation (6), we get

$$\begin{aligned}\underline{\nabla}\,\underline{u} &= \left(\frac{\partial u_r}{\partial r}\underline{e}_r + \frac{\partial u_\theta}{\partial r}\underline{e}_\theta + \frac{\partial u_z}{\partial r}\underline{e}_z\right)\otimes\underline{e}_r + \frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}\underline{e}_r + u_r\underline{e}_\theta + \frac{\partial u_\theta}{\partial\theta}\underline{e}_\theta + u_\theta(-\underline{e}_r) + \frac{\partial u_z}{\partial\theta}\underline{e}_z\right)\otimes\underline{e}_\theta \\ &+ \left(\frac{\partial u_r}{\partial z}\underline{e}_r + \frac{\partial u_\theta}{\partial z}\underline{e}_\theta + \frac{\partial u_z}{\partial z}\underline{e}_z\right)\otimes\underline{e}_z.\end{aligned} \tag{8}$$

We have gotten two extra terms here due to change in basis vectors. We can use the above equation to obtain the displacement gradient matrix in cylindrical coordinate system. The coefficient of the basis tensor $\underline{e}_i \otimes \underline{e}_j$ goes into i^{th} row and j^{th} column of the matrix to finally yield the following:

$$[\underline{\nabla}\,\underline{u}]_{(r,\theta,z)} = \begin{matrix} \\ \text{r} \\ \theta \\ \text{z}\end{matrix}\begin{matrix} \begin{matrix} r & \qquad\theta & \qquad z\end{matrix} \\ \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial u_r}{\partial\theta} - u_\theta\right) & \dfrac{\partial u_r}{\partial z} \\ \dfrac{\partial u_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial u_\theta}{\partial\theta} + u_r\right) & \dfrac{\partial u_\theta}{\partial z} \\ \dfrac{\partial u_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_z}{\partial\theta} & \dfrac{\partial u_z}{\partial z}\end{bmatrix}\end{matrix} \tag{9}$$

1.4 Representation of Strain tensor (start time: 16:09)

Now, we can use equation (1) to obtain strain matrix which is the symmetric part of the displacement gradient matrix derived above. It turns out to be the following:

$$\left[\underline{\underline{\epsilon}}\right]_{(r,\theta,z)} = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{2}\left[\dfrac{1}{r}\left(\dfrac{\partial u_r}{\partial \theta} - u_\theta\right) + \dfrac{\partial u_\theta}{\partial r}\right] & \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) \\ \dfrac{1}{2}\left[\dfrac{1}{r}\left(\dfrac{\partial u_r}{\partial \theta} - u_\theta\right) + \dfrac{\partial u_\theta}{\partial r}\right] & \dfrac{1}{r}\left(\dfrac{\partial u_\theta}{\partial \theta} + u_r\right) & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) & \dfrac{\partial u_z}{\partial z} \end{bmatrix} \tag{10}$$

If we compare this with the strain matrix in Cartesian coordinate system given in equation (2), we can notice extra terms here.

2 Physical significance of strain components (start time: 19:14)

2.1 Significance of ϵ_{rr} (start time: 19:20)

$$\epsilon_{rr} = \left(\underline{\underline{\epsilon}}\,\underline{e}_r\right) \cdot \underline{e}_r = \frac{\partial u_r}{\partial r} \tag{11}$$

This is called radial longitudinal strain or *radial strain* simply. To visualize this, we can think of a typical cross section of a hollow cylinder as shown in Figure 2.

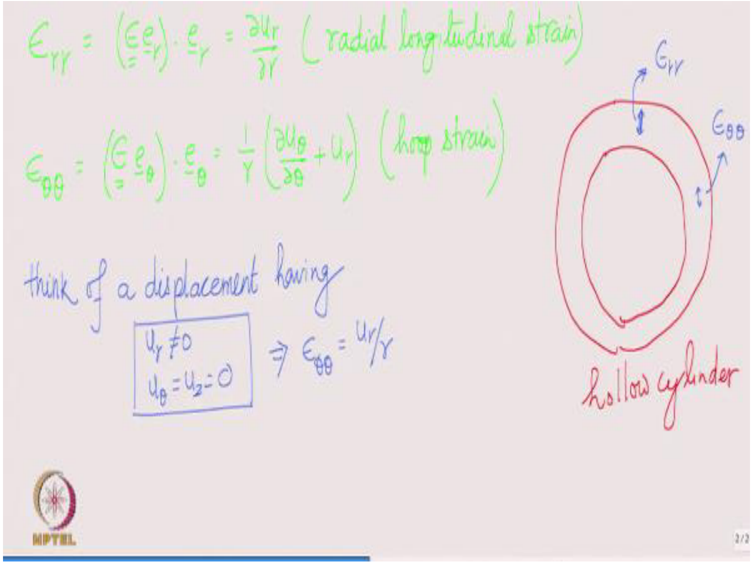

Figure 2: A typical cross-section of a hollow cylinder with longitudinal strains for two line elements, one in the radial direction and the other in the θ direction also shown.

The elongation of a radial line element gives us ϵ_{rr} as shown.

2.2 Significance of $\epsilon_{\theta\theta}$ (start time: 21:03)

$$\epsilon_{\theta\theta} = \left(\underline{\underline{\epsilon}}\,\underline{e}_\theta\right)\cdot\underline{e}_\theta = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) \tag{12}$$

This strain is also called *hoop strain* or *circumferential strain*. This is the elongation of a line element directed along θ direction (circumferential line element) as shown in Figure 2. The circumferential strain has two contributions. The partial derivative term is intuitive because longitudinal strain along a direction is understood as the derivative of displacement in that direction with respect to the same direction. The other term $\frac{u_r}{r}$ is the unusual term which we now try to understand physically. Think of a displacement which has only radial component, i.e.,

$$u_r \neq 0, \quad u_\theta = 0, \quad u_z = 0 \tag{13}$$

For such a displacement, if we find $\epsilon_{\theta\theta}$ using equation (12), we will get

$$\epsilon_{\theta\theta} = \frac{u_r}{r} \neq 0 \tag{14}$$

Figure 3 again shows a typical cross section of a hollow cylinder. For the displacement given in (13), all points in the cross-section simply displace radially.

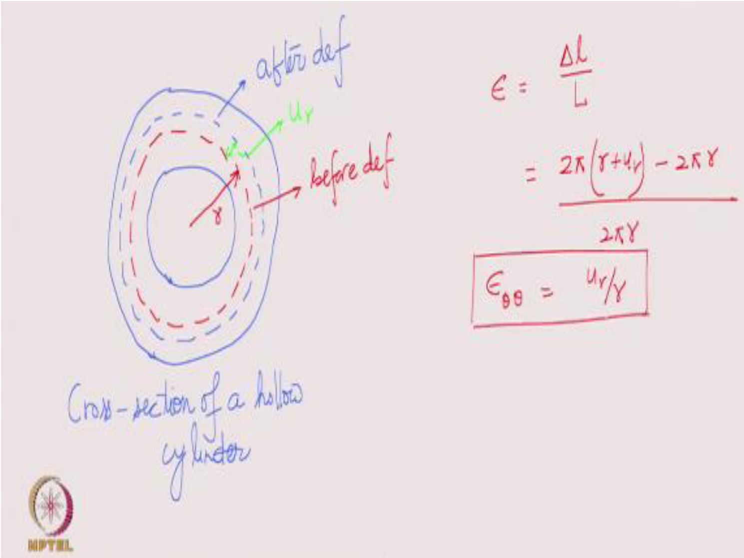

Figure 3: A typical cross section of a hollow cylinder: the circumferential lines for both reference and deformed configurations are shown for the displacement function in (13)

We have also drawn a circumferential line both before and after deformation. All points on this line initially at radial coordinate r displaces to radial coordinate $r+u_r$. We can notice that the length of the original circumferential line (shown in red) has increased generating longitudinal strain in it, i.e.,

$$\epsilon = \frac{\Delta l}{L} = \frac{2\pi(r+u_r) - 2\pi r}{2\pi r} = \frac{u_r}{r} \tag{15}$$

which by definition is $\epsilon_{\theta\theta}$. This specific case helps us to visualize the extra term present in the formula for hoop strain. Despite u_θ being zero, the extra term generates non-zero $\epsilon_{\theta\theta}$.

2.3 Significance of $\gamma_{r\theta}$ (start time: 27:19)

We can see from the strain matrix that we have an extra term in $\epsilon_{r\theta}$ also, i.e.,

$$\gamma_{r\theta} = 2\epsilon_{r\theta} = 2\left(\underline{\underline{\epsilon}}\,\underline{e}_r\right) \cdot \underline{e}_\theta = \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial r} \tag{16}$$

This denotes the change in angle between two initially perpendicular line elements directed along $\underline{e}_r$ and $\underline{e}_\theta$. Figure 4 shows a typical cross-section of a cylindrical body. At an arbitrary point, we consider two line elements directed along $\underline{e}_r$ and $\underline{e}_\theta$ respectively. You should try to figure out the physical meaning of the extra term that we get in $\gamma_{r\theta}$.

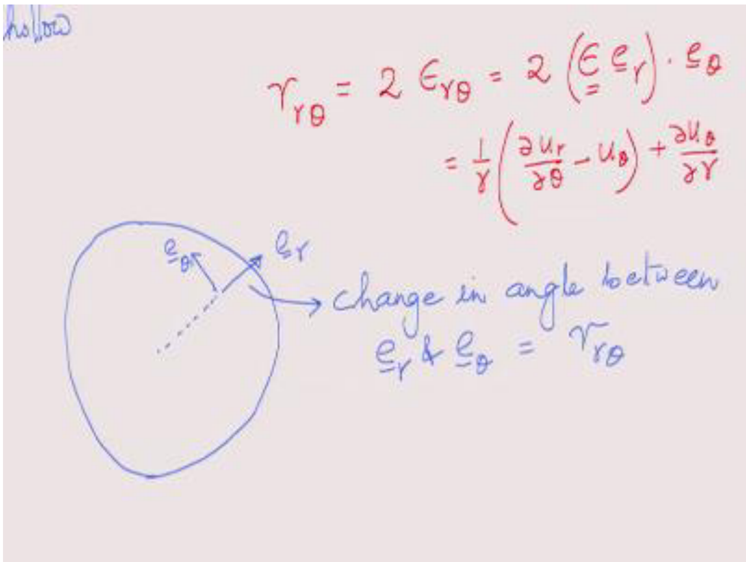

Figure 4: Two line elements are considered at a point on the cross section of a cylindrical body directed along $\underline{e}_r$ and $\underline{e}_\theta$

2.4 Significance of other components (start time: 29:37)

The other strain components have no unusual term. The quantity γ_{rz} gives us shear strain between line elements along $\underline{e}_r$ and $\underline{e}_z$, $\gamma_{\theta z}$ gives us shear strain between line elements along $\underline{e}_\theta$ and $\underline{e}_z$ and finally, ϵ_{zz} gives us longitudinal strain of a line element directed along $\underline{e}_z$.

3 Relating stress and strain in cylindrical coordinate system for isotropic materials (start time: 30:08)

Once we have stress and strain matrices in cylindrical coordinate system, let us relate them for an isotropic material. We know how to relate stress and strain in Cartesian coordinate system. We also know that for an isotropic material, all material properties are independent of the direction. Thus, the relationship between stress and strain components must also be independent of the coordinate system. This means that we could choose any set of three perpendicular directions and resolve our stress and strain tensors in those directions but the mathematical form of their relationship would remain unchanged. For example:

$$\epsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})) \Rightarrow \epsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})) \tag{17}$$

We can obtain all other relations in a similar way leading to

$$
\begin{aligned}
\epsilon_{\theta\theta} &= \frac{1}{E}(\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})), \\
\epsilon_{zz} &= \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})), \\
\gamma_{r\theta} &= \frac{\tau_{r\theta}}{G}, \\
\gamma_{rz} &= \frac{\tau_{rz}}{G}, \\
\gamma_{\theta z} &= \frac{\tau_{\theta z}}{G}.
\end{aligned}
\tag{18}
$$

We emphasize that the above relationship would have a different mathematical form if the material were not isotropic.

Having obtained stress and strain components and their relation in cylindrical coordinate system, we will learn in the next few lectures how using them for deformation of cylindrical bodies leads to simplified form of equations. Such problems could also be solved in Cartesian coordinate system but the equations would not simplify then.