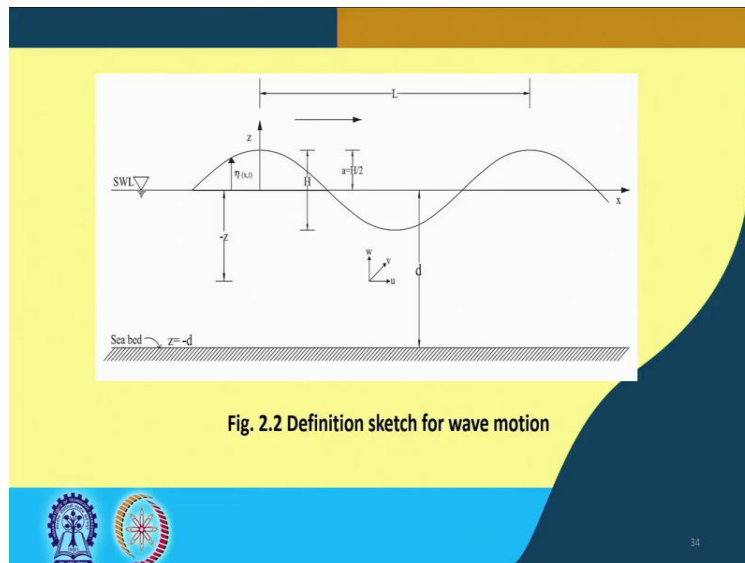


Hydraulic Engineering
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Lecture # 61
Introduction to wave mechanics (Contd.)

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Welcome back students for this third lecture of this module, where we are deriving the velocity potential of the ocean surface waves. Last time we concluded with this particular slide indicating the definition sketch of the wave motion, where we have told you what the water depth is this figure is actually very important for understanding how this linear wave is represented. The wavelength is given as L at the top the depth is indicated wave height H is also given no sea bed is shown so we from this point onward will continue with our derivation.

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SOLUTION TO THE LAPLACE EQUATION:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (2.6)$$

- Method of separable is used to obtain the solution to Eqn. (2.6)
- Let us assume

$$\phi(x, z, t) = \bar{X}(x)\bar{Z}(z)\bar{T}(t) \quad (2.7)$$

Now we have listed the boundary conditions one was $w = 0$ at the bottom second was we specified free surface dynamic free surface boundary condition at the top. And now we start with the solution of the Laplace equation. So, Laplace equation in terms of velocity potential in 2 dimension, as we have assumed x and z is $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$.

To obtain the solution a very simple method of separable is used to obtain the solution to this equation which is 2.6. Our let us assume, main method of separable ϕ of x, z, t can be written as X bar as a function of x it can be separated Z bar as z , T bar as t .

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- Substituting Eq. (2.7) in Eq. (2.6) we get

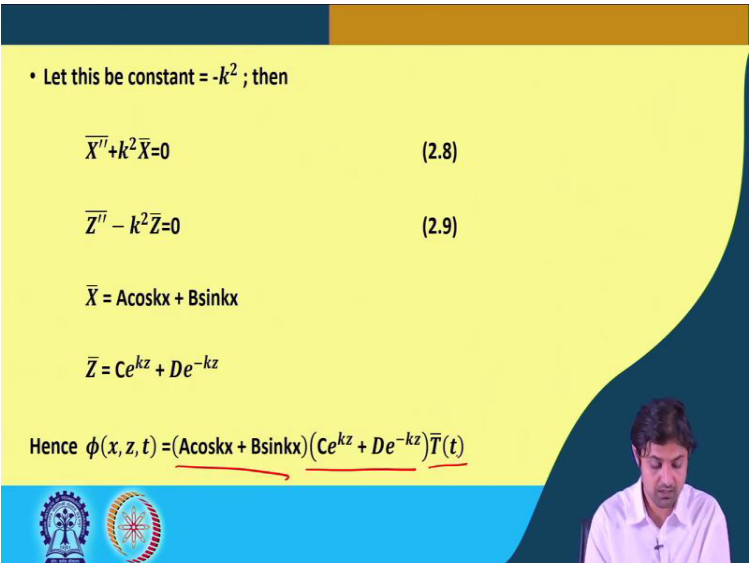
$$\bar{X}'' \bar{Z} \bar{T} + \bar{X} \bar{Z}'' \bar{T} = 0$$

- Where each prime denotes differentiation once with respect to the particular independent variable.
- Dividing both side of the above $\bar{X}\bar{Z}\bar{T}$ gives

$$\frac{\bar{X}''}{\bar{X}} = -\frac{\bar{Z}''}{\bar{Z}}$$

Now, if we substitute 2.7 this equation into 2.6 because due to the velocity potential Laplace equation properties, we are able to separate X, Z and T using the method of separable what do we get x double bar. So, x double prime bar Z bar and T bar + X bar Z double prime whole bar and T bar = 0. Here each prime denotes differentiation once with respect to the particular independent variable for double means double differential. Now, if we divide both sides by X bar, Z bar and T bar, we get X bar X double prime bar by X bar = - Z double prime whole bar by said this is the equation that we get.

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• Let this be constant = $-k^2$; then

$$\bar{X}'' + k^2 \bar{X} = 0 \quad (2.8)$$

$$\bar{Z}'' - k^2 \bar{Z} = 0 \quad (2.9)$$

$$\bar{X} = A \cos kx + B \sin kx$$

$$\bar{Z} = C e^{kz} + D e^{-kz}$$

Hence $\phi(x, z, t) = (A \cos kx + B \sin kx) (C e^{kz} + D e^{-kz}) \bar{T}(t)$

Now, see this is we let us say this is a constant - K square, then we can write 1 of the equations from this will come X double prime bar + k squared x bar = 0 and the second one will be Z double prime bar - k square z bar = 0, these 2 are very famous equation the solution for this is standard. So, x bar from 2.8 can be written as A cos k x + B sine k x. Whereas Z from equation number 2.9 solution to this type of equation is here solution to the equations of type 2.9 is this one from equation 2.9.

We get X bar and Z bar so, we can simply write so, you see we had the solution that we all initially assumed was phi x, z, t is X bar, Z bar and T bar so we can write instead of X bar we can write this one instead of Z bar we can read this one and T bar remains as T bar.

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$$\bar{X} = A \cos kx + B \sin kx$$

$$\bar{Z} = C e^{kz} + D e^{-kz}$$

Hence $\phi(x, z, t) = (A \cos kx + B \sin kx)(C e^{kz} + D e^{-kz})\bar{T}(t)$





Now, the solutions to ϕ are simple harmonic in time, that is, because we know from previous that they are waves and periodic in time. Therefore, $\bar{T}$ can be simply replaced as $\cos t$ or $\sin t$ anything it can be replaced, and therefore with each of the values of $\sin t$ or $\sin \sigma t$, we have 4 forms of ϕ one will be $A \cos kx$ multiplied by $\sin$, the other would be $A \cos kx$ multiplied by $\cos$. The other would be $B \sin kx$ multiplied by $\sin$ and then the third the fourth one would be $B \sin kx$ multiplied by $\sin$.

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• The solutions to ϕ are simple harmonic in time requiring $\bar{T}(t)$ be replaced as $\cos(\sigma t)$ or $\sin(\sigma t)$, thus leading to four forms of solutions to ϕ , such that

- $\phi_1 = A_1 (C e^{kz} + D e^{-kz}) \cos(kx) \cdot \cos(\sigma t)$
- $\phi_2 = A_2 (C e^{kz} + D e^{-kz}) \sin(kx) \cdot \sin(\sigma t)$
- $\phi_3 = A_3 (C e^{kz} + D e^{-kz}) \sin(kx) \cdot \cos(\sigma t)$
- $\phi_4 = A_4 (C e^{kz} + D e^{-kz}) \cos(kx) \cdot \sin(\sigma t)$



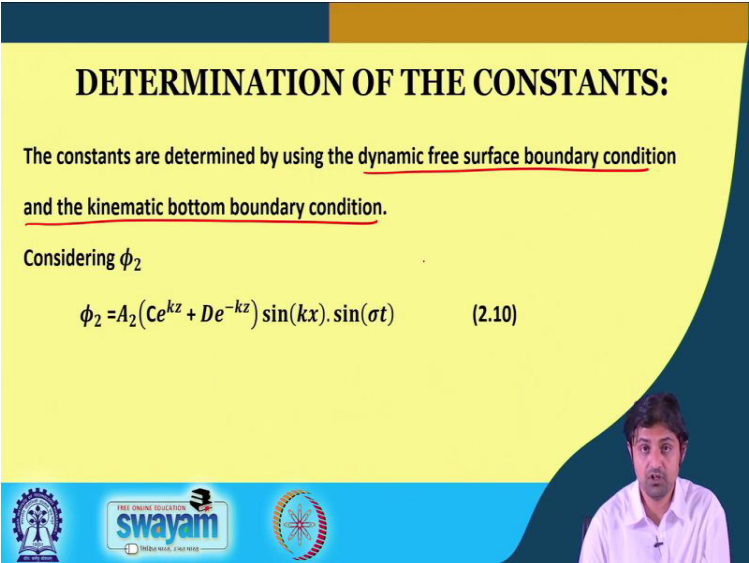


So we can write 4 different terms A_1 we did not touch this, so $A \cos kx + B \sin kx$ multiplied by either $\sin \sigma t$ or $\cos \sigma t$. There are going to be 2 terms when we assume t as a function of $\sin \sigma t$, and they would not mean after multiplication with this and there will be 2

other term is if instead of sin we assume cos t bar T . So total we can write 4 terms, ϕ_1 will be $A_1 \cos kx \cos \sigma t$. Another thing would be $A_2 C e^{\text{to the power } k z} \cos kx \sin \sigma t$. $+ D e^{\text{to the power } -k x} \sin kx \sin \sigma t$.

The third would be $\sin kx \cos \sigma t$ and fourth would be $\cos kx \sin \sigma t$. However the total velocity potential is going to be some of 2 of those which one we will come to it later. But for now, we have assumed 4 different terms $\phi_1 \phi_2 \phi_3 \phi_4$, 1 would be $\phi_1 - \phi_2$ the other could be $\phi_3 - \phi_4$, but that will come to later.

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DETERMINATION OF THE CONSTANTS:

The constants are determined by using the dynamic free surface boundary condition and the kinematic bottom boundary condition.

Considering ϕ_2

$$\phi_2 = A_2 (C e^{kz} + D e^{-kz}) \sin(kx) \sin(\sigma t) \quad (2.10)$$

The slide also features logos for Swayam and the Indian Institute of Space Science and Technology (IIST) at the bottom, and a small video inset of a man speaking in the bottom right corner.

So, we have to see this these terms $\phi_1, \phi_2, \phi_3, \phi_4$ are constants A_1, C, D, σ is related to the time period case rate to the wave period that will come to it later. But more importantly we have to find these constants A_1, C and D . So, how did it remind these constants, the constants are determined using the dynamic free surface boundary condition and the kinematic bottom boundary condition as you would remember, because, by the definition of the boundary condition, we said that, there could be infinite number of solutions to a governing equation?

However, with the application of boundary condition, we can find one or more unique solutions to the problem that is what we are going to do. These constant will be determined from the boundary Conditions let us consider ϕ_2 is one velocity potential that we have written here this

one. So, what does it say ϕ_2 is A_2 into $C e$ to the power $kz + D e$ to the power $-kz$ into $\sin kx$ into $\sin \sigma t$ we call this equation number 2.10.

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• Applying the kinematic bottom boundary condition

i.e., $\frac{\partial \phi}{\partial z} = 0$ at $z = -d$

$$\left. \frac{\partial \phi_2}{\partial z} \right|_{z=-d} = A_2 (Cke^{-kd} - Dke^{kd}) \sin(kx) \cdot \sin(\sigma t) = 0$$

$A_2 \neq 0; \sin(kx) \cdot \sin(\sigma t) \neq 0$ [since velocity potential exists]

$\therefore C = De^{2kd}$

If we apply the kinematic bottom boundary condition, which means fields the velocity potential ϕ is given by $\frac{\partial \phi}{\partial z}$ and that will be 0 at $z = -d$. If you apply that $\frac{\partial \phi}{\partial z}$ at $z = -d$ it will be A_2 into $C e$ to the power $-kd - D k e$ to the power kd into $\sin kx \sin \sigma t = 0$ at $z = -d$. So, now there is no Z but instead of Z we have d . So, A_2 is not equal to 0 otherwise there is not going to be any wave also $\sin kx \sin \sigma t$ will not be 0 because there will be a velocity potential if we substitute them to be either one of them to be 0 the velocity potential becomes 0.

So that means these 3 terms cannot be 0. Therefore, the only thing is that this be 0 and if we utilize this we are going to get C as D into e to the power kd . So with one boundary condition we have reduced our unknown by 1 number. $C = D e^{2kd}$ we have related C and D .

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Substituting for C in (Eq. 2.10) and simplifying,

$$\phi_2 = 2A_2 D e^{kd} \left(\frac{e^{k(d+z)} + e^{-k(d+z)}}{2} \right) \sin(kx) \cdot \sin(\sigma t)$$

$$\phi_2 = 2A_2 D e^{kd} \cosh k(d+z) \sin(kx) \cdot \sin(\sigma t) \quad (2.11)$$


Now if we substitute for C in equation 2.10 so instead, of CV put 2d into what the term that we have got we are going to get $\phi_2 = 2A_2 D e^{kd} \sin kx \sin \sigma t$ or simply we can write this is nothing but $\cosh k(d+z)$. Simple trigonometry.

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$$\left. \frac{\partial \phi_2}{\partial t} \right|_{z=0} = 2A_2 D \sigma e^{kd} \cdot \cosh kd \cdot \sin(kx) \cdot \cos(\sigma t)$$

On assuming

$$\eta = a \sin(kx) \cdot \cos(\sigma t) \text{ where } a = \text{wave amplitude} = \frac{H}{2} \text{ and applying the free}$$

surface boundary condition $\left\{ \eta = \frac{1}{g} \frac{\partial \phi_2}{\partial t} \right\}_{z=0}$ we get

$$\frac{1}{g} \frac{\partial \phi_2}{\partial t} \bigg|_{z=0} = \eta$$


So, we have got $\phi_2 = 2A_2 D e^{kd} \cosh k(d+z)$ so we are left with A_2 and D for now. And if we do $\frac{\partial \phi_2}{\partial t}$ at $Z = 0$. We get $2A_2 D \sigma e^{kd} \cosh kd \sin kx \cos \sigma t$ and on assuming if we assume that free surface elevation is $A \sin kx \cos \sigma t$. So, what we are going to do here is the wave amplitude that is $H/2$ this is our assumption and by applying the free surface boundary condition that $\eta = \frac{1}{g} \frac{\partial \phi_2}{\partial t}$ at $Z = 0$.

So, first we are applied the bottom boundary condition. On the second step we want to apply the dynamic free surface boundary condition. So, for applying the dynamic free surface boundary condition we need to calculate this term one by $\nabla^2 \phi$ at $Z = 0$. And this is what we get this term and this we equate to η is the free surface which $= A \sin kx \cos \sigma t$.

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$$a \sin(kx) \cos(\sigma t) = \frac{2A_2 D \sigma e^{kd}}{g} \cosh(kd) \sin(kx) \cos(\sigma t)$$

$$\frac{2A_2 D \sigma e^{kd}}{g} = \frac{ag}{\sigma} \frac{1}{\cosh(kd)}$$

Substituting in eq. (2.11), we get

$$\phi_2 = \frac{ag \cosh(k(d+z))}{\sigma \cosh(kd)} \sin(kx) \sin(\sigma t) \quad (2.12)$$

So simply what we do $\eta =$ and after solving for this we get $2A_2 D e^{kd}$ is ag by σ into 1 by $\cosh(kd)$. And if we substitute these values into our original ϕ_2 we are going to get it as $\phi_2 = ag$ by σ $\cosh(kd + z)$ into $\sin(kx)$ into $\sin(\sigma t)$. So, we have now, first what we did we solve for Laplace equation applied the 2 boundary conditions and obtain the value of the velocity potential ϕ_2 which is of the form ag by σ into $\cosh(kd + z)$ into $\sin(kx)$ into $\sin(\sigma t)$.

So, σ is 2π by T and k is 2π by L this is wavelength and this is time period for you the derivation is not important what the results are, but the derivation is important in a way that it makes you appreciate how starting from the basics we can derive these equations they will ask the potential.

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