Solid Mechanics
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Lecture - 24
Bending of beams (contd.)

Hello everyone! Welcome to Lecture 24! We will continue with our discussion on bending of beams. In the previous lecture, we learnt about pure bending of beams in which bending moment was constant along the beam. In this lecture, we will discuss non-uniform bending of beams.

1 Non-uniform Bending (start time: 00:43)

1.1 Introduction (start time: 00:58)

In case of pure bending, we had the same moment acting on every cross-section. For this reason, pure bending is also called uniform bending. We will now move on to non-uniform bending of a beam where bending moment is not uniform along the length of the beam. In the previous lecture, we had derived the following relation for bending moment M and curvature κ:

$$M = EI\,\kappa. \tag{1}$$

If bending moment M is not constant along the length, bending curvature will also not be constant. We now consider another case of loading where we can also have distributed load $b(x)$ acting on the beam as shown in Figure 1.

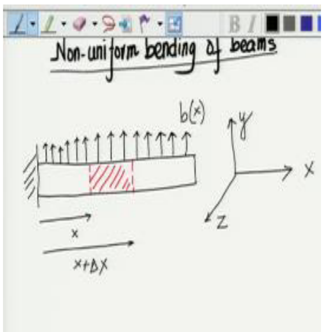

Figure 1: A beam acted upon by a distributed load $b(x)$

The load $b(x)$ is assumed to act in $+y$ direction. Let us cut a small part of the beam of length Δx at a distance x from the clamped end and (see the red region in Figure 1) and further draw its free body diagram as shown in Figure 2.

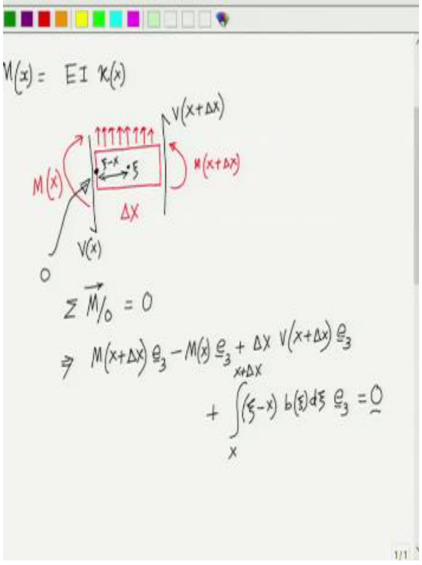

Figure 2: Free body diagram of a small part of the beam shown in red in Figure 1.

On this section, apart from distributed load, there will be bending moment as well as shear force acting at the two ends. By convention, for the cross section having normal in $+x$ direction, shear force acting in $+y$ direction is considered as positive while for the cross section having normal in $-x$ direction, shear force acting in $-y$ direction is considered as positive. We denote this shear force by V. The center of the left cross-section is marked as O and the position of a general point in the section of the beam is denoted by ξ which varies from x to $x + \Delta x$. Due to static equilibrium, the net moment on this part of the beam must be zero about any point. In particular, let us do moment balance about O:

$$\sum \overrightarrow{M}/_O = \underline{0}$$

$$\Rightarrow \quad M(x+\Delta x)\ \underline{e}_3 - M(x)\ \underline{e}_3 + \Delta x\ V(x+\Delta x)\ \underline{e}_3 + \int_x^{x+\Delta x} (\xi - x) b(\xi) d\xi\ \underline{e}_3 = \underline{0} \tag{2}$$

As everything is pointing in $\underline{e}_3$ direction, we can easily extract $\underline{e}_3$ component of the equation to get the following scalar equation:

$$M(x+\Delta x) - M(x) + \Delta x\ V(x+\Delta x) + \int_x^{x+\Delta x} (\xi - x) b(\xi) d\xi = 0 \tag{3}$$

Since this equation holds for an arbitrary part of the beam having arbitrary length Δx, we can divide both the sides by Δx and take the limit $\Delta x \to 0$ to shrink the section to a single point, i.e.,

$$\lim_{\Delta x \to 0} \frac{M(x+\Delta x) - M(x) + \Delta x\ V(x+\Delta x) + \int_x^{x+\Delta x} (\xi - x) b(\xi) d\xi}{\Delta x} = 0$$

$$\Rightarrow \quad \frac{dM}{dx} + V(x) + \lim_{\xi \to x} (\xi - x) b(\xi) = 0 \qquad (\because \text{as } \Delta x \to 0,\ \xi \to x)$$

$$\text{Or,} \quad \frac{dM}{dx} + V(x) = 0. \tag{4}$$

We have derived an important relation between the variation of bending moment and shear force. It says that whenever moment varies along the beam, there has to be a non-zero shear force acting on the beam's cross-section. The case of zero shear force corresponds to the case of pure bending where moment is constant throughout the length of the beam as we saw in the last lecture.

1.2 Variation of σ_{xx} in the cross-section (start time: 11:20)

For the case of non-uniform bending, the variation of σ_{xx} can be taken to be the same as in the last lecture. We just have to use local bending moment in the formula, i.e.,

$$\sigma_{xx}(x, y, z) = \frac{-M(x)y}{I_{zz}} \tag{5}$$

In the above expression, y represents distance from the neutral axis as earlier.

2 Variation of τ_{yx} in the cross-sectional plane (start time: 12:07)

The shear components of traction in the cross sections are τ_{yx} and τ_{zx}. As there is an overall shear force $V(x)$ acting on the cross section in y direction, τ_{yx} must be non-zero. Let us now try to obtain its distribution in the cross-section.

2.1 Assumption (start time: 13:40)

The stress component τ_{yx} in a cross section can be a function of both y and z in general. However, we make a simplifying assumption that it is only a function of y and not of z. This means that τ_{yx} would be the same at all points on lines parallel to z axis as shown in Figure 3 - different horizontal lines will have different τ_{yx} though.

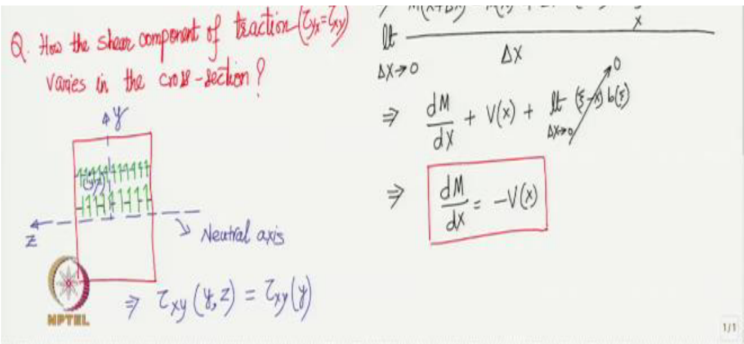

Figure 3: A typical cross section of the beam with variation of τ_{yx} shown such that it is a function of y alone

2.2 Analysis (start time: 16:44)

To find the distribution of τ_{yx}, we cut a small cuboid element from the beam as shown in Figure 4 in green.

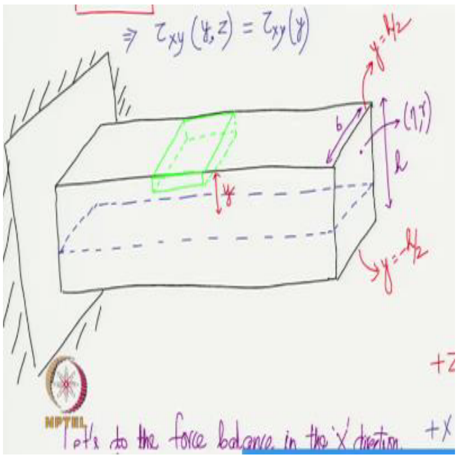

Figure 4: A cuboid element cut from the beam.

A zoomed view of this green cuboid with all external loads acting on it is shown in Figure 5.

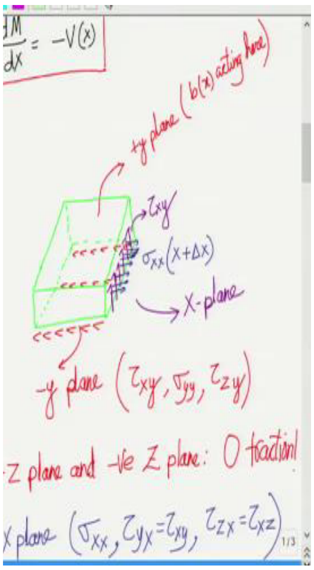

Figure 5: Free body diagram of the cuboidal element cut from the beam shown in Figure 4.

The bottom surface ($-y$ plane) of this element is at a distance of y from the neutral plane where as its left face ($-x$ plane) is at a distance x from the beam's clamped end. Its free body diagram is shown in Figure 5. The top face is the $+y$ plane where the distributed load $b(x)$ acts. The bottom face is the $-y$ plane and traction components τ_{xy}, σ_{yy} and τ_{zy} act on it in $-x$, $-y$ and $-z$ directions, respectively. The $+z$ and $-z$ faces (side faces) are part of the lateral surfaces of the original beam. As external forces are assumed to be applied only on the $+y$ plane of the beam, the $+z$ and $-z$ faces of the element are traction free. On the $+x$ face, we have bending stress σ_{xx} that we have derived already. We also have τ_{yx} and τ_{zx} acting there. Similarly, σ_{xx}, τ_{yx} and τ_{zx} act on $-x$ plane but in negative directions. In order to find τ_{yx}, we just need to balance the forces on this small cuboidal element in x direction. Let's first consider the force due to σ_{xx} on the $+x$ plane. We also introduce coordinates ξ, η and γ for x, y and z variations from the centroid of the cross-section of the beam at the clamped end. Thus, any general point on the cross section of the beam has (y,z) coordinates as (η,γ) as shown in Figure 4. The total force due to σ_{xx} on the $+x$ face of the small cuboidal element will be obtained by its integration over the area of this face. As evident from Figure 4, η varies from y to $\frac{y}{2}$ and γ varies from $\frac{-b}{2}$ to $\frac{b}{2}$ on $+x$ and $-x$ faces of the element. Thus, force due to σ_{xx} on the $+x$ and $-x$ faces will be

$$\int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{y}^{\frac{h}{2}} \sigma_{xx}(x+\Delta x, \eta, \gamma)\ d\eta d\gamma \quad - \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{y}^{\frac{h}{2}} \sigma_{xx}(x, \eta, \gamma)\ d\eta d\gamma. \tag{6}$$

The $+z$ and $-z$ faces of the element being traction free and do not contribute to force in any direction. The force on $+y$ face has no component in x direction since the external distributed load acting there is assumed to act along y direction. However, $-y$ face has τ_{xy} acting on it which contributes to force in x direction. The local coordinate ξ varies from x to $x + \Delta x$ while γ varies from $\frac{-b}{2}$ to $\frac{b}{2}$ for y faces of the element. Thus, the net force due to τ_{xy} in x direction will be

$$\int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{x}^{x+\Delta x} -\tau_{xy}(\xi, y, \gamma) d\xi d\gamma. \tag{7}$$

Summing all the forces in x direction to zero, we get

$$\int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{y}^{\frac{h}{2}} \Big[\sigma_{xx}(x+\Delta x, \eta, \gamma) - \sigma_{xx}(x, \eta, \gamma)\Big] d\eta d\gamma \quad - \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{x}^{x+\Delta x} \tau_{xy}(\xi, y, \gamma) d\xi d\gamma = 0 \tag{8}$$

We can also substitute the following expression for σ_{xx}

$$\sigma_{xx} = \frac{-M(x)\eta}{I_{zz}(x)} \tag{9}$$

in equation (8), which yields

$$\iint_{x \text{ plane}} \left[\frac{-M_z(x+\Delta x)\,\eta}{I_{zz}(x+\Delta x)} - \frac{-M_z(x)\,\eta}{I_{zz}(x)}\right] dA \quad - \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{x}^{x+\Delta x} \tau_{xy}(\xi, y, \gamma) d\xi d\gamma = 0. \tag{10}$$

As the first integral is over x planes, M_z and I_{zz} act as constants. To make things simpler, we assume that the beam's cross section does not vary along its length. Thus, I_{zz} does not vary along the length. Similarly, we assumed initially that τ_{xy} does not vary with z or γ coordinate. All these simplifications lead to

$$-\left[\frac{M_z(x+\Delta x) - M_z(x)}{I_{zz}}\right] \iint_{x \text{ plane}} \eta dA \quad - \int_{x}^{x+\Delta x} \tau_{xy}(\xi, y) d\xi \ \int_{\frac{-b}{2}}^{\frac{b}{2}} d\gamma = 0. \tag{11}$$

As the above equations holds for cuboid element of any length Δx, we can shrink its length such that Δx approaches zero. As always, we first divide the above equation by Δx and then take limit $\Delta x \to 0$, i.e.,

$$
-\left[\lim_{\Delta x \to 0} \frac{M_z(x+\Delta x) - M_z(x)}{\Delta x}\right] \frac{1}{I_{zz}} \iint_{x \text{ plane}} \eta dA \quad - b \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} \tau_{xy}(\xi, y) d\xi}{\Delta x} = 0
$$

$$
\Rightarrow \quad -\frac{dM_z}{dx} \frac{1}{I_{zz}} \iint_{x \text{ plane}} \eta dA \quad - b \; \tau_{xy}(x,y) = 0
$$

$$
\Rightarrow \quad \frac{V(x)}{I_{zz}} \iint_{x \text{ plane}} \eta dA \quad - b \; \tau_{xy}(x,y) = 0 \qquad \text{(using (4))} \tag{12}
$$

Notice the simplification in second integral. As $\Delta x \to 0$, the range of the second integral shrinks to point x itself. So, the integrand $\tau_{xy}(\xi,y)$ becomes a constant, i.e., $\tau_{xy}(x,y)$ and comes out of the integral which is then multiplied by the length of the integration interval Δx and further divided by Δx. The integral in the first term above is y-moment of x face of the cuboid element which is also shown as the shaded area in Figure 7.

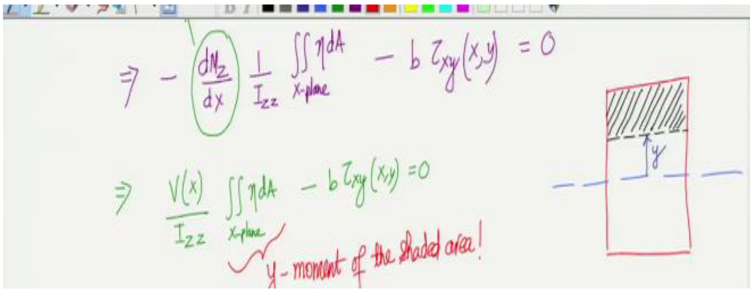

Figure 7: The cross section of the beam with the shaded region representing the area of the x face of the small cuboidal element.

We denote this moment by $Q(y)$: a function of y alone. As we change y, the shaded area changes and thus the first moment of the shaded area also changes. Thus, the final expression for τ_{xy} becomes

$$
\boxed{\tau_{xy}(x,y) = \frac{V(x)Q(y)}{I_{zz}\, b(y)}} \tag{13}
$$

which is same as τ_{yx}, the shear component in the cross-sectional plane. Here, we have allowed b, the width of the cross section, to vary with y. This enables us to use this result for cross-sections of beams such as I-beams. If we compare equations (5) and (13), we see that while bending stress σ_{xx} is proportional to moment, shear stress τ_{yx} is proportional to shear force.

3 Variation in τ_{yx} for some representative cross-sections (start time: 43:40)

3.1 Rectangular cross-section (start time: 43:40)

A typical rectangular cross section is shown in Figure 8 and we want to find the value of τ_{yx} at a distance of y from the neutral axis.

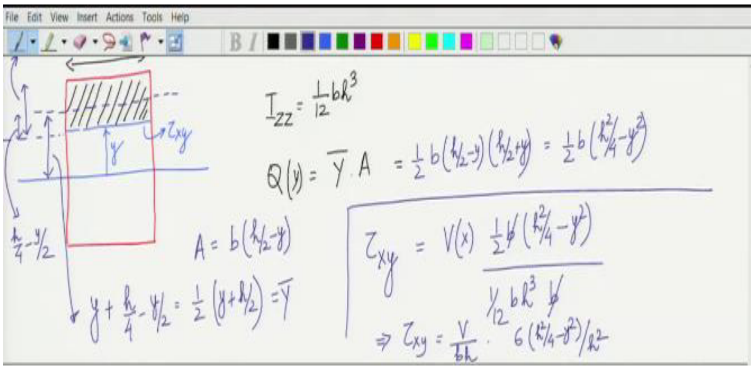

Figure 8: A rectangular cross section for the calculation of τ_{yx}.

As τ_{yx} is independent of z, it will have the same value over lines parallel to the z axis. For applying equation (13), we need to find $Q(y)$, $b(y)$ and I_{zz}. As this is a rectangular cross section, width $b(y)$ is constant and equal to b. We have already derived I_{zz} for a rectangular cross section in one of the previous lectures to be

$$I_{zz} = \frac{1}{12}bh^3 \tag{14}$$

We only need to obtain expression for the first moment $Q(y)$ of the area above y line where τ_{yx} is to be calculated (shown as the shaded region in Figure 8). The first moment will simply be y coordinate of the centroid of the shaded area multiplied by the shaded area. As the height of the shaded area is $\frac{h}{2}$ - y, its centroid will be at half of this distance from the y line and hence at

$$\bar{Y} = y + \frac{1}{2}(\frac{h}{2} - y) = \frac{1}{2}\left(y + \frac{h}{2}\right) \tag{15}$$

from the neutral axis. Thus, $Q(y)$ becomes

$$Q(y) = \bar{Y}A = \frac{1}{2}\left(y + \frac{h}{2}\right)b\left(\frac{h}{2} - y\right) = \frac{1}{2}b\left(\frac{h^2}{4} - y^2\right) \tag{16}$$

while τ_{yx} becomes

$$\tau_{yx} = \frac{V(x)\ \frac{1}{2}b\left(\frac{h^2}{4} - y^2\right)}{\frac{1}{12}bh^3\ b} = \frac{V}{bh} \times 6\left(\frac{1}{4} - \left(\frac{y}{h}\right)^2\right) \tag{17}$$

As V is the total shear force on the cross section and bh is the area of the cross section, $\frac{V}{bh}$ equals average shear stress τ_{avg} while τ_{yx} at the neutral axis is

$$\tau_{yx}(y = 0) = \tau_{\text{avg}} \times 6\left(\frac{1}{4} - 0\right) = \frac{3}{2}\tau_{\text{avg}}$$

Likewise, at the periphery of the cross section $\left(y = \pm\frac{h}{2}\right)$, τ_{yx} is

$$\tau_{yx}(y=0) = \tau_{\text{avg}} \times 6\left(\frac{1}{4} - \frac{1}{4}\right) = 0 \tag{18}$$

The variation of y vs τ_{yx} is shown in Figure 9.

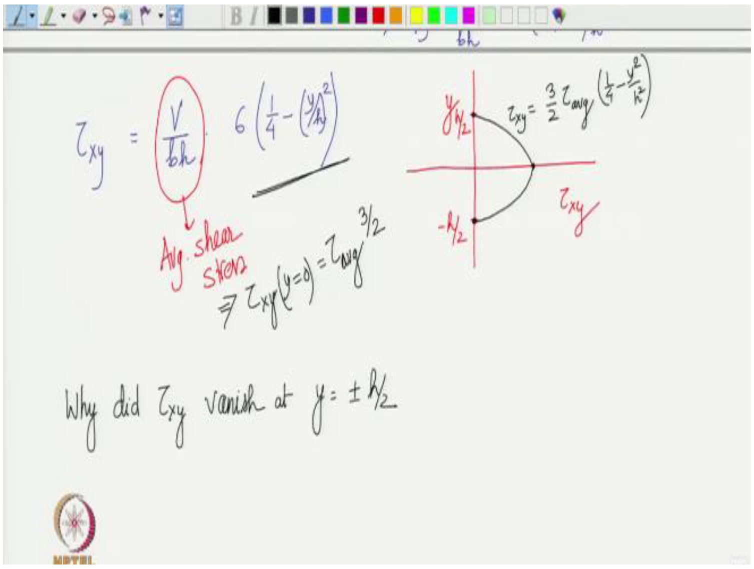

Figure 9: Plot of y vs τ_{yx} for a rectangular cross section

We observe that due to the presence of shear force in the cross section, shear stress is maximum at centroid and vanishes at the two ends. There is another way to realise the vanishing of shear stress at the ends. The points $y = \frac{h}{2}, \frac{-h}{2}$ also lie on top and bottom surfaces of the beam, respectively. There is no external traction on the bottom surface whereas on the top surface, the distributed load $b(x)$ acts in y direction. Thus, τ_{xy} is zero at both top and bottom surfaces. However, due to τ_{xy} and τ_{yx} being equal, shear stress on cross-sectional plane vanishes at $y = \frac{h}{2}, \frac{-h}{2}$.

3.2 Circular cross-section (start time: 54:30)

In the derivation for rectangular beams, we had assumed that τ_{yx} is independent of z coordinate. For a circular beam however, this assumption cannot be used. Consider the beam shown in Figure 10 and analyze one of its cross-sections.

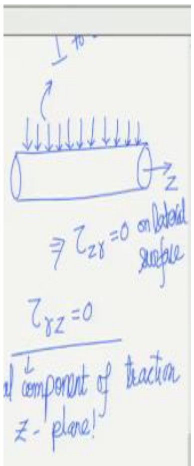

Figure 10: A radial distributed load applied on a circular beam

If τ_{yx} is constant along lines parallel to z axis, the shear stress will be as shown in Figure 11.

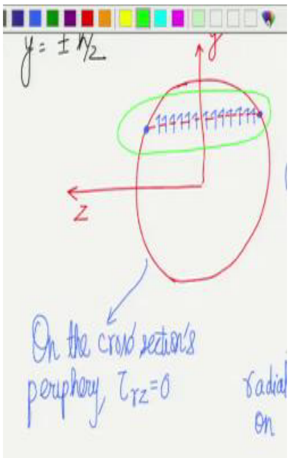

Figure 11: τ_{xy} on a cross section of a circular beam with the assumption that τ_{xy} is independent of z.

Basically, it is non-zero even at the ends. Let us work with cylindrical coordinate system and assume a radial distributed load is acting (e.g., pressure load) which could also be zero. In that case τ_{zr} must be zero on the lateral surface. So, τ_{rz} must also be zero along the periphery of the cross section, i.e., shear stress cannot have radial component along the periphery of the cross-section in the cross-sectional plane. However, if we look at Figure 11, the assumption of τ_{yx} being independent of the z coordinate leads to a non-zero radial component which is a contradiction. Thus, we can conclude that for circular cross-sections, considering τ_{yx} independent of z is not a good assumption. Still, this assumption is often used since it gives an approximate distribution of shear stress.

3.3 I-beam cross-section (start time: 59:45)

Figure 12 shows the cross-section of an I-beam.

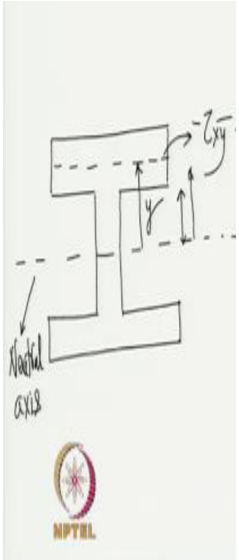

Figure 12: The cross section of an I-beam

The centroid of this section would be at the center because of symmetry. So, the neutral axis passes through the center. There are two different values of width possible in the cross section. To find τ_{yx} at a distance y from the neutral axis, we can again use equation (13). As the width changes abruptly in this case, the distribution of shear stress will also exhibit a jump corresponding to this abrupt change in width b. A plot of y vs. τ_{yx} is shown in Figure 13 exhibiting this jump.

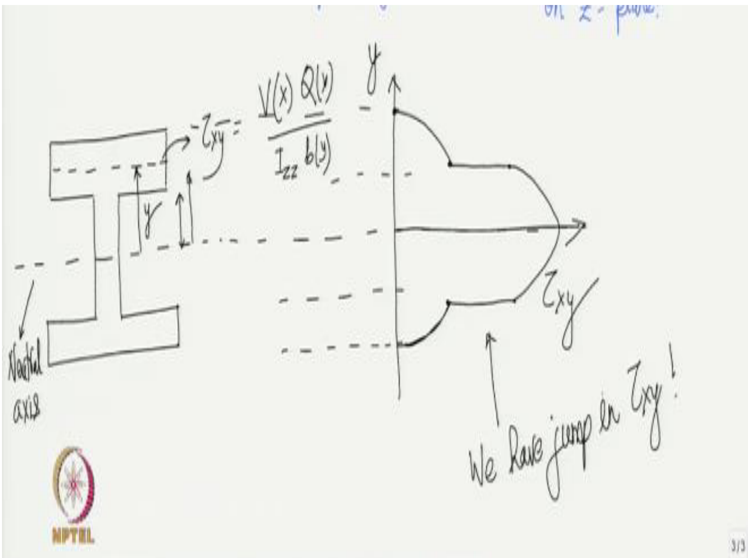

Figure 13: Plot of y vs τ_{yx} for an I-section.

This concludes our discussion on non-uniform bending of beams. Till now, we have only looked at beams having symmetrical cross-section. In the next lecture, we will learn how the analysis differs for beams having asymmetrical cross-section.