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Lec 25: Mass Conservation Equation- II

Let us start on the continuity equations. The basically mass conservation equations, mass conservations for infinitely small control volumes as we discuss it that means the control volumes what we consider it, its the dimensions dx dy dz they are very very close to the 0. So when you have the control volumes infinitely small in that case we are looking at how we can have a differential equations format for mass conservation equation. As we derive it the basically when you have a basic components like we are looking for rho is a density field which is a scalar quantities functions of positions x, y, z in a Cartesian coordinates, then you have the time component. Then we have the time component, v is the velocity field. which is having a scalar component of smaller u, v and w which will be the functions of positions x, y, z, t.

So for considering that through the Gauss theorem concept we try to created basic equations in this format. That is what indicating for us there is a two component in this basic equations. Now one of the component is if you look at this part in any control volumes you have change of the mass components per unit volume. That is what is coming as root by T.

This is the components indicating for us change of the mass storage within the control volume. That is what will be equal to the divergence of the max flux within the control volume. This is what we derived from the Gauss theorems for a control volumes. We can get it. This is what the change of the mass within the control volume per unit volume.

This is what net outflux of the mass going through this control surface that is what we are represent as a divergence of mass flux the rho v is the divergence of the mass flux is equal to the 0. Now let us go to the next level instead of looking this vector rotations we can look at a small infinitely small control volumes okay which is can be a simplified a parallel piped or box dimensions where we can put this the coordinate of x, y, z and along this the x directions we have a dx, dy and the dz. So this is what our control volumes, this is what the coordinate the Cartesian coordinates. Let us start deriving these mass conservation equations for infinite small control volumes, infinitely small control volumes. As we are discussing it, we are going to get the same form of equations what we have derived from the Gauss theorems.

But here to understand more clearly, we are taking a simplified control volumes. we are considering a simplified a parallel piped control volumes which is having a dimensions of dx, dy and the dz. The velocity fields are the u, v, w. So velocity fields is u, v, w. As you know it the velocity is a functions of space and the time.

So, for these simple control volumes we want to derive what could be the mass conservation equations. The same we have to look at the change of the mass storage within the control volume locate the net outflux of max flux going through this control surface. The advantage of these ones, this case we have considered a very complex control volumes as well as the control surface. In order to not going for very complex vector multiplicate divergence fields and all we just consider a control volume where you have a dimensions of smaller dx dy dz and you have the plane which is a perpendicular to the all these Cartesian coordinates. So that is the easy component what we have done it.

Now if I just derive this part as it is a functions of positions and the time the velocity components then what I can do it as I say that the density is also the functions of positions and the time density is also the positions of space and time. Here the very basic things is that I can apply the Taylor series which it is a very basic concept that any functions if it is a continuous variable then we can approximate using Taylor series expansions. Using the Taylor series expansions we can approximate it. So basically as I consider at the centroid of this box the velocity is equal to u v and w what will be the velocity at the different faces that is what we are going to look it there is a front face, the rear face, top, bottom, left side and the right sides. So if you look at this is having 6 faces this is what we are having the 6 faces for these 6 faces we are deriving this part.

Now if you look it if I put a Taylor series expansions okay. I am looking the Marx flux as you know it the Marx is rho u into area rho into q that is what is a Marx flux that is what is if I looking in the x directions then rho u will come it. what will be the rho u value at the right face which is dx by 2 distance apart from the center that is the distance is dx by 2. So if I apply the Taylor series I will get it at the center the right face of the centers I will have a velocity the first term the second terms and third terms like this I will have a infinity series. I will have the infinity series from the Taylor series expansions as I have the velocity field at sorry the mass flux is rho u at this point what will be the max plus at the right face that is what will be commit from the Taylor series that is any functions you can apply the Taylor series and you can get this the gradient component at different first gradient, the second gradient and these terms.

Now if you look at this Taylor series components, if as we are looking for a infinitely small control volumes, so the dx value what we are considering defined by the characteristics length. Basically when you are solving a pipe flow the characteristics length is the diameters. So if you have the dx by dl for any general problems this value can come to a very very smaller value of 10 to the power minus 3 that is the range okay. So dx by L because the we are considering the control volume which is a infinitely small as compared to the characteristics length this can be 1 by 1000 times okay so 10 but when you like just square it dx by L if I square it this is what will be 10 to the power minus 6. So if you look at these terms that if I going for a first component, second component, third component like this, this component can be negligible.

because as giving the explanations with a characteristic length if I square it it will further it will be a neglected component. Similar way all the terms are will be negligible as compared to the first two terms. So we use in the Taylor series expansion for these derivations only these two component all other components we neglected. So that is the basic component and if you look it that way what will be the velocity component at the center of rear face. That means we are getting the velocity component at this point okay so that is what we got it from Taylor series.

The same way if I look at the velocity component of the rear face that way I can write it in a z directions so in a z directions we can have a components and if you look at these components, the center of the top surface. So this is what will come it as the points. So if you look at all the components of the at the surface what is the mass flux enters at the point of the centroid with a centroid locations if I try to look it whatever the velocity component will come it as I can get it that velocity components will come it is a this is what at the dx by 2 distance far away. So that is what I will get it that is what mass plus coming into this control volume which is a dx by 2 distance from the centroid so I will get this component. I will get this component which will be a positive this side and the negative this side.

The same way if I look it along the y directions I will have a positive this component which is I just considering the first two terms of the Taylor series and I will have a negatives in this side. So that way if you look it this is what max flux in these directions coming into the control volume in this direction is going out this is coming in going out. So very simple way we are writing this mass flux components from the centroid we where we have a the velocity component of u v w. So, we are we can able to write it on all these six spaces as it is given it center bottom surface you will have this component that way if I proceed it I can get it the components. Now if you look it what I am looking for these control volumes is the mass conservations.

First I am to look at that the rate of change of the mass conservations within the control volume. That is what I can define it in this term. The rate of change of mass conservations, the V stands for volume rho into dV and doing a volume integrals, I will get the mass conservations. the control volume which is infinitely small, I can simplify it. This is what is equal to the dx dy dz is the volume and this integral part I can just simplify it to make it this form.

because it is a infinitely small control volumes because infinitely small control volume we can approximate this volume integrals to in this form okay. This is what rate of the change of the mass within the control volume. Now if you look it how much of mass flux is coming in and going out. So if I put it all the terms okay this is what mass flux as any surface if I consider it okay let me I consider this is the x direction surface the surface the surface area will be dy dz the surface area. The velocity mass flux is a perpendicular to that.

So because of that I can get the mass flux will be the rho times of velocity and the area dy dz okay. So this is a very basic concept we are using it to estimate the mass flux. the getting the surface area multiplying with a rho into the u that is what is going it mass plus coming it. So this u value is a varying it this rho u values we have estimated at different surface.

So we know the inflow max. also the outflux mass. That is what is presented here. So if you look at net mass flow rate into these control volumes how much mass flux coming into that is what is this component, this component as well as you can understand in the z direction. So this is the x directions component This is the y directions component, this is the z directions component and you can just look it the surface area which is varying from here as I was explaining to you dy by dz the in the y direction the surface area will be dx by dz. Z generations you will have a surface area will be dx into dy.

This is what the mass net mass flow rate into the control volume. So same way we can write it mass outflux. How much of mass is going out from this control volume? So if you look at this mass outflow from this control volume that is what you will commit this component. As you can easily remember this because outflux part we have the positive components okay. You have a rho u and gradient of rho u into dx by dt rho v gradient of rho v in y directions dy by dt dy by 2 rho w and you will have the gradient of rho w dz by t and this is the surface area.

So you can remember it just to visualize that is a max flux going out from this control volumes which is having a dx by 2D stands, dy by 2D stands and dz by 2D stands in x, y, z directions respectively. This is the point is lies from the centroid. So that is the reasons as we consider the rho u, rho v, rho w is a continuous functions okay. We have considered the rho u rho v rho w are the continuous functions. And we also looking at that using

Taylor series considering only the first two terms.

other terms we are neglected then we can expand it for a infinitely small control volume what will be mass influx, mass outflux. As we discuss it that we can write a very simple way the net outflux just subtracting that part we will get a net outflux mass flux okay. Net max fluxes this is the net the change of the storage of the mass flux with the control volumes. This is the mass flux in x directions. This is the mass flux, the net mass flux in y directions.

This is the mass flux in z directions. So, if you look at this way it is a same simple equations we are deriving it considering this part that rho u rho v rho omega is a continuous functions Taylor series valid only we are considering the first two terms then we can equate it right with a simple forms this is what our equations will commit and you can easily cancel this part. dx dy dz is the volume of this control volumes that is what you can cancel out and you can write in the forms which already we are discussing the Cartesian coordinates as it is indicating the rho by rho t is a mass flux change of the mass storage this is the mass fluxes in the three directions per unit volume of the control volume. Per unit volume of the control volumes that is what is the equations are representing it. So, physically you can understand it that we are looking for change of the mass storage within the control volume with respect to time.

The net mass flux is going through the x directions. Net mass flux in x directions. This is what is net mass flux in y directions and this is what in z directions. So if you look at that with a smaller control volume, infinite smaller control volumes, we can also derive the same equations what we have derived from Gauss theorems. And I can always write it in a compact forms.

as if you look it I am not going to detail derivations to making more complexity here but I will just explaining you that what the basic equations we have got it that rho t plus in vector forms I can write it dot is rho V vectors is equal to 0. So these equations if I expand these terms and do some arithmetic operations finally I will get it. these terms and these terms. If you look at that part, what I am looking at is that if I try to very examples wise giving this part. We are looking at the change of the density in material element is a negligible small compared to magnitudes of the density and that what if this value is equal to 0 that is what we will have this velocity divergence is equal to the 0 which is approximate as a incompressible flow.

Let me I put it in examples when you will be derive it I am just looking it not looking at detailed derivations of that part who are really interested they can look at the Senzel Cimbala book for detailed derivations of that part. Now let me look at many of the problems like for examples we have the problems like cylindrical coordinate systems. where we can solve the problems with a cylindrical coordinate systems. The same way we can solve the problems with a cylindrical coordinate systems. Here what we are looking it that as why we define the velocity field u, v, w in Cartesian coordinates for cylindrical coordinate system, cylindrical coordinate systems we define the velocity field as ur is a radial velocity component u theta this is the velocity components in the angular directions you can look it okay this is radial point angular point and you have the perpendicular to that is the uz representing r theta and z.

So many of the problems we can use a cylindrical coordinates to solve it because it will be easier for imposing the boundary conditions also we can solve the problems. That means earlier we have considered a parallel piped control volumes instead of taking the parallel pipe control volumes we are considering a control volumes which is having a radial d dimensions of dr d theta and perpendicular to this will be the dr d theta and perpendicular will be the dz. So as you consider with a as the cylinder coordinates elements and the basically you know it the convergence from cylinder coordinates with the Cartesian coordinate systems having the relationship with R 10 theta and the XY. that is what is the relationship here that is what is a simple the graphically if you look at that x will be the r cos theta, y will be r sin theta and theta will be the tan theta equal to y by x. So our Cartesian coordinate systems we are looking converting to the cylinder coordinate systems.

Then I am not going for derive of this but also we can get it the similar form of equations. If you look at that again we can get it similar form of equations with u r, u theta and u z and there are the terminology to look at that point is there. So this is also saying it same concept is change each mass fluxes in a radial and theta direct tangential directions and the z directions that what will be equal to the change of these components. So, please try to physically interpreted these terms and then try to basically you can try to remember these equations or you can try to understand each physical components as we have derived for Cartesian coordinates. then we are also derived for the c-linear coordinate.

Let us come back to understand some physical interpretations of these equations. Now I will have some case studies. For examples you consider the case which is the steady compressibles. As the problem says that it is it is a steady compressible okay. So as this it is a steady problems all this time derivative components will be the 0 okay.

So because it is a steady problems because does not change with the time. The velocity fields, the density field does not change with the time. If you have that conditions any partial derivative with respect to times that is what will give to us 0. So because of that if you look at the equations component we have same part where I am just writing it that I have divergence of rho V vector is equal to 0.

okay. So if I have the steady problems okay I am coming back to these equations. So if I have a steady problems please remember this equation do not try to remember other components because it is quite easy to derive other components if I know the basic component of these equations. So as a students please try to remember these equations part okay then other components we can derive for the Cartesian coordinate systems. as I have the steady compressible flow that means density is a function of positions of it does not have a dependency with the time okay and if this is my basic equations then it is a very easily we can say this component is 0 as it is a steady problems okay as it is a steady problems this component is 0 only you have the divergence of this component okay. these components okay and you try to remember it the divergence is the del is del u by del x i del y del y j unit vector del z k vectors.

So this is what you know from basic mathematics. So if you look at these components any dot products any the divergency of the mass fluxes that product if you do it as this component is 0 you can easily find out this component. is it correct so easily you can get it this component just you are multiplying it okay so just trying to write it again to explaining these terms this is just a vector dot products that del is del by del x i del by del y j del by del z k vector okay. u v you can see that u v vectors you can write it u rho u i rho v j and rho w k. So you know it, if i into i will be 1, i into j equal to 0, i into k equal to 0, j into i equal to dot product of j and i unit vectors will be 0, j into j will be 1 and k into k will be the 1, so just products. So if you just make a dot product of these two things, you will get this part.

That is what my point is please try to remember these components then you can try to understand it how the components are coming it for the Cartesian coordinate systems. Now if you look at this Cartesian coordinate systems I have this components which is coming for the steady flow is this part. if you look at these components as I am explaining to you that for a steady compressible flow steady compressible flow what we are getting it the basic equation is. 0. What do you mean by that? If you have the steady compressible flow where there is no change of the mass storage as that means if you have any flow domain if you have this steady compressible flow any flow domains what we are getting it there will be no change of the mass here.

for at any instant if there is a change of this velocity field that is what you will be indicating for means how many source or sinks are there. How many source or the sink S1, S2, S3, S are there because rate of the change must is 0. There is no change of the mass within this control volume as it is a steady problem. So, there will be a summation of the source and sinks that should be equal to 0.

That is what is showing by these divergence vectors. Now, if you look at this incompressible flow, if you go for nest is the incompressible flow what very interesting things why get it for incompressible flow incompressible flow that means density is a constant okay so as the density is a constants you can understand it we have this part will be 0 and this is a constant component will be come out. So, we will get a very simple way that velocity divergence is equal to 0. That is what if a coordinate Cartesian coordinate formats if I put it I will get this component. Now let us interpret it what it actually happened.

incompressible and the compressible flow. That means in case of the compressible flow without steady or unsteady behavior that is does not matter it if it is a incompressible flow that means if there is a any change in this control in a velocity field that what will be propagate very instantly throughout the domain. I am just repeating it. If you have the incompressible flow that is what is most of the time we encounter it. So, in case of the incompressible flow so the since there is a no time components here So, even if the unsteady flow if there is a any change of the velocity field that is what will affect the throughout the domains that is what will be affect in a throughout the domains. But in case of the compressible flow that is what is representing here.

If case of the compressible flow like loud noise as equivalent to a bomb blast and all. So what will happen it, it will move the disturbance what will be created in this velocity field that is what will move a marching it. That is what will be the shock wave marchings will happen it in case of incompressible flow. sorry in case of compressible flow but incompressible flow cases we if there is a change of the velocity field that is what will felt it throughout the flow domains. Whereas in case of compressible flow you can see there will be a shockwave propagations like a very strong noise and all that is what will create a shockwave propagations it is moving from one point and propagating radiately out from this that is what you can see it and you can interpret it between these two.

flow systems. Before concluding this I can say that please try to remember these two equations and try to interpret it how the flux components are coming it and what is each components indicating for us that should be understand it then it is maybe mathematically looks very complex but it is a very simplified to understand the mass flux component and change of the mass storage component. Let us solve very two simple problems before ending this class. Let us take a one questions is from Sinzel Simbala book okay we can refer it. The case is there is air-fuel mixture compressed by a piston.

This is if you can look it. There is a piston is moving it in an internal combustion engine.

The origins of y coordinates at the top of the cylinders. That is what is here.

Top of the cylinder. Y is point straight down as shown. Piston assumed to move up. This is piston is moving up at a constant speed of v phi. v phi is the constant speed. The distance L is between the top of the cylinders and the positions as it moves it decrease with the time with a linear approximations of this form.

That is what is showing it. So the distance that is what is L bottom minus of v phi t. So velocity into time that is the distance okay. So that means this L decreasing as a functions of time and v phi is the velocity of the pistons. If you have a t equal to 0 the density of the air mixture is everywhere is constant that is rho 0 at the time equal to 0. this is a rho naught then estimate the density of the fuel mixtures as a functions of time given the parameters during this pistons of stokes.

So if you look at these problems it looks very complex but it is a simple problems using the mass conservations we are looking it for internal commonsense engines which is a quite interesting problems which is there in Sinzel Simbala book. Now if you look it what are the assumptions you have to do it that is what you have to try to understand it what we are looking it we are looking the row okay which is functions of time okay that is what we are looking it functions of times and we have this assumptions is this okay. So density varies with time does not vary with a space that means we are not considering within the pistons okay. So here we are considering a density is a constant. So these variations of the density we are not considering in a space only the as this is a pistons is a functions of the time.

the space variations we are not considering it that is what we are assuming it is uniform in the cylinder only changes with the time that is the assumptions you can clearly write it now if you look it the problems is only the y directions here okay so there is other directions is not there so we can always write it the velocity in the y directions is a functions of y and t. Other velocity components like x directions and the z directions can consider it 0 because this is a one-dimensional nature of the problems. This is the one-dimensional nature of the problems which is we simplified it. Now we are getting a two assumptions. One is regarding this density variations that is what we can consider is only a functions of time.

Then we can consider the velocity components only the V component will have a functions of y and t there is no dimensions of x and z directions we are not considering that part. So, other velocity components can be also consider is 0. If it is that And there is no mass escapes from these cylinders.

That is why this is very clear cut. So, because there is no mass escapes. This piston is moving. So, there is no mass escaping from this. So, as it is a compressions are happening it.

Now, look at the solutions. What we are looking at the solutions part. Let me I explaining that part. First we are looking it as this cylinder is moving it, this is the y directions. So, we can consider the velocity variations within these pistons area, the vertical velocity can consider as a linear functions. can consider a linear functions that means at y equal to 0 the velocity is 0. At y equal to L we have the velocity is equal to minus Vp that is what is the velocity variations what is happening it L is a length is a function of time.

So if you look at that now we are looking a compressibles. So we have the compressible continuity equations as we discuss it that I can write this continuity equations this form okay which is now as I have rho W is a 0 components. So these are all will be the 0. So it can easily give us this relationship. Now as the density is not a functions of y. If you just look at these problems how we are simplifying the functions at different levels.

if you consider the density as not a functions of y otherwise its problems could be the very complex problems. We consider density is not a functions of y that is what it help us that we can come out this row from this y variability and if I put it this form just rearrange it I will get this part. I will get this part and if I do the integrations then finally I will get this relationship. I am not going more detail the differentiations and simplifications part.

More details you can go through this book where is it is available there. Now if you look it if it is consider it next slides if you look it. if I consider this because anyway rho is only functions of time so we can convert from partial derivative to total derivatives and I can integrated this part finally I get the relationship of these components. The log components are there. So we have the expressions of the density which will be functions of time, the length of the pistons, L bottoms and if I do a non-dimensional things, I will get it this form.

So I define it as a non-dimensional form. with a rho non-dimensional rho by rho at the zeroth time. V star also I am putting a non-dimensional form. So I am getting this form. See if I plot this part okay in a non-dimensional form okay.

So T star is starting from 0 okay. go up to 1 and this is the value go up to the 3. If it is that there I will get a relationship functions up to 0.8 will have a something like this you can just plot it. So what it says that if you try to physically interpreted these problems whenever you solve the problems more interesting is there to understand the physical component of the problems instead of only looking this mathematical component. When you just look at that when you have a L bottom is equal to 0 that means the case will come it that means if you distance moves it. go to the zero level okay that possible will not possible so in the problems because that is what will be the infinite to come.

So density will come as a infinite value that is the reasons most of this international internal combinations engines we have a 20 percent is a left close to the 20 percent for clearance volumes. That is what is this piston moves up to that. That is what is feasible. We cannot make it density is infinities.

As you were compressing it, we cannot make it for a real case is a density infinity. That is what you can see from this figure. So it will go up to 0.8 and you will have a density which will be the increase almost the 3 times of whatever density value. So many of the times using this mass conservation equations really we can solve many interesting problems like here how the density variations and in a internal combustion engines where the resistance is moving it and also we are talking about what practical conditions comes it.

So more details you can go through Sindhal Simbala book. Now if you look at another problems which is a similar to problems in many of the gate or engineering service exams where they give this velocity field okay. As you can see it there is a two-dimensional velocity field unsteady two-dimensional velocity field is given you have to verify it whether the flow field can be as approximate as as a incompressible flow which is very easy questions and most of the times the gate engineering service also look at this type of questions where they have given the velocity field okay. Only there is a angular frequency components are there to find out whether the flow is incompressible or not. As we discuss it flow is incompressible means you have this is what equal to 0 and since it is a 2-dimensionals we have a further simplifications. So, we will just substituting the u and v scalar velocity components and check it whether this becomes 0.

So, others are the steps as the solution starts it this is a u components and the v components you can look it this is a u component and v component. you are doing this partial derivatives part as I just discussing it this component is not there. So you just do the partial derivative with respect to X and the Y then you get it the value is 0.

80 then it is a 0. So that means this what becomes 0. So flow is That is why it is very easy questions with this let us complete today class just to have a summarize it if you can look at mass continuity concept with a alternative forms. This is what I have not detailed derived it, we can go through that. This is very basic equations as we have written so

many times to repeat this. With this, let me conclude this lectures giving a thank to my students group who have been helping us to prepare this slide. Thank you. Thank you.