Chapter 28: Linear Transformations

Introduction

Linear transformations are a cornerstone of linear algebra, playing a critical role in the mathematical formulation and analysis of engineering problems. In civil engineering, they are used in structural analysis, finite element methods, and computer-aided design, among many other applications. A linear transformation provides a systematic way of mapping vectors from one vector space to another, preserving the operations of vector addition and scalar multiplication. This chapter explores the theory and properties of linear transformations, matrix representations, and their applications with a particular focus on geometrical intuition and problem-solving methods relevant to engineering contexts.

28.1 Definition of a Linear Transformation

A **linear transformation** (or linear map) is a function $T: V \to W$, where V and W are vector spaces over the same field F, such that for all $u, v \in V$ and all scalars $c \in F$:

$$1.\ T(u+v) = T(u) + T(v) \text{(Additivity)}$$

$$2.\ T(cu) = cT(u) \text{(Homogeneity)}$$

These two properties ensure that linear transformations preserve the linear structure of vector spaces.

28.2 Examples of Linear Transformations

1. **Identity Transformation**:

$$T(x) = x, \forall x \in R^n$$

2. **Zero Transformation**:

$$T(x) = 0, \forall x \in R^n$$

3. **Scaling Transformation**:

$$T(x) = \lambda x, \lambda \in R$$

4. **Rotation in R^2**:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

5. **Projection onto a Line or Plane**

28.3 The Matrix of a Linear Transformation

If $T: R^n \to R^m$ is a linear transformation, then there exists a unique matrix $A \in R^{m \times n}$ such that:

$$T(x) = Ax, \forall x \in R^n$$

This matrix is called the **standard matrix** of the linear transformation. If the basis of the domain and codomain is standard, then:

$$A = [T(e_1) T(e_2) \ldots T(e_n)]$$

where e_i are the standard basis vectors of R^n.

28.4 Kernel and Image of a Linear Transformation

Kernel (Null Space):

The **kernel** of T, denoted by $ker(T)$, is the set of all vectors in V that are mapped to the zero vector in W:

$$ker(T) = \{v \in V \mid T(v) = 0\}$$

It is a subspace of the domain V.

Image (Range):

The **image** or **range** of T, denoted by $\mathrm{Im}(T)$, is the set of all vectors in W that are images of vectors in V:

$$\mathrm{Im}(T) = \{T(v) \mid v \in V\}$$

It is a subspace of the codomain W.

28.5 Rank and Nullity

For a linear transformation $T: R^n \to R^m$, the **rank** of T is the dimension of its image, and the **nullity** is the dimension of its kernel.

The **Rank-Nullity Theorem** states:

$$\dim(\ker T) + \dim(\operatorname{Im} T) = \dim(V)$$

Or in terms of matrices:

$$\text{nullity}(A) + \text{rank}(A) = n$$

This theorem is crucial for analyzing the solvability and behavior of linear systems.

28.6 Composition of Linear Transformations

If $T_1: U \to V$ and $T_2: V \to W$ are linear transformations, then their composition $T_2 \circ T_1: U \to W$ is defined by:

$$(T_2 \circ T_1)(u) = T_2(T_1(u))$$

Properties:

- The composition of two linear transformations is also a linear transformation.
- If T_1 and T_2 have matrix representations A and B, respectively, then:

$$[T_2 \circ T_1] = BA$$

28.7 Invertible Linear Transformations

A linear transformation $T: V \to W$ is **invertible** if there exists another linear transformation $S: W \to V$ such that:

$$S \circ T = I_V, T \circ S = I_W$$

In terms of matrices:

- If $A \in R^{n \times n}$ is the matrix of T, then T is invertible iff $det(A) \neq 0$, and the inverse transformation is represented by A^{-1}.

28.8 Geometrical Interpretation of Linear Transformations

Linear transformations in R^2 and R^3 can be visualized as operations such as:

- **Rotation**
- **Reflection**
- **Scaling**
- **Shearing**
- **Projection**

These transformations can change the orientation, length, or position of vectors while preserving linearity. Civil engineers often encounter such operations in structural modeling, mechanics, and computer simulations.

28.9 Linear Transformations and Systems of Linear Equations

A system of linear equations can be viewed as a linear transformation:

$$A x = b \Rightarrow T(x) = b$$

- The solution exists iff $b \in \operatorname{Im}(T)$
- The solution is unique iff $ker(T) = \{0\}$

This perspective is fundamental in understanding the solvability and structure of linear systems in applied engineering contexts.

28.10 Applications in Civil Engineering

- **Structural Analysis**: Displacement and force transformations in trusses and frames.
- **Finite Element Method (FEM)**: Transformation of stiffness matrices.
- **Coordinate Transformations**: Switching between local and global coordinate systems.

- **CAD Modelling**: Rotations, scaling, and projections used in designing components.
- **Stress-Strain Relations**: Linear mappings between stress and strain tensors in elasticity theory.

28.11 Change of Basis and Similarity of Matrices

Linear transformations can be represented differently depending on the **basis** used for the vector space. This concept is crucial in simplifying problems or interpreting data from different reference frames.

Change of Basis

Let $T: V \to V$ be a linear transformation and suppose $B = \{v_1, \ldots, v_n\}$ and $B' = \{v'_1, \ldots, v'_n\}$ are two different bases for V. Let P be the **change of basis matrix** from B to B'. Then:

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This transformation of matrix representations under different bases is called **similarity**.

Similarity of Matrices

Two matrices A and B are **similar** if there exists an invertible matrix P such that:

$$B = P^{-1} A P$$

Similar matrices represent the **same linear transformation** under different bases. They have:

- The same determinant
- The same trace
- The same characteristic polynomial and eigenvalues

28.12 Eigenvalues and Eigenvectors of Linear Transformations

An important class of linear transformations are those that **scale** vectors instead of changing their direction.

Given a linear transformation $T:V \to V$, a non-zero vector $v \in V$ is called an **eigenvector** of T if:

$$T(v) = \lambda v$$

for some scalar $\lambda \in F$, which is called the **eigenvalue** corresponding to v.

Finding Eigenvalues and Eigenvectors

Let A be the matrix of the linear transformation T. The eigenvalues satisfy:

$$det(A - \lambda I) = 0$$

This is called the **characteristic equation**. Solving it gives the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. For each λ_i, the eigenvectors are found by solving:

$$(A - \lambda_i I)x = 0$$

Importance in Civil Engineering

- **Modal Analysis**: In structural dynamics, eigenvalues represent natural frequencies of vibration.
- **Principal Directions**: In stress analysis, eigenvectors correspond to principal stress directions.
- **Stability Analysis**: Eigenvalues indicate the stability of equilibrium in systems modeled by differential equations.

28.13 Diagonalization of Linear Transformations

A square matrix $A \in R^{n \times n}$ is **diagonalizable** if there exists an invertible matrix P and a diagonal matrix D such that:

$$A = P D P^{-1}$$

This is equivalent to saying that the linear transformation has **n linearly independent eigenvectors**.

Conditions for Diagonalizability

- Matrix A has n distinct eigenvalues ⇒ always diagonalizable.
- If not all eigenvalues are distinct, check for linearly independent eigenvectors.

Geometrical Meaning

Diagonalization simplifies the transformation into **scalings along specific directions** (eigenvectors). For example, in a vibrating beam, diagonalization simplifies coupled motion equations into independent modes.

28.14 Linear Operators and Matrix Powers

A **linear operator** is a linear transformation $T : V \to V$ on a single vector space. Matrix powers of linear operators are useful in recurrence relations, system modeling, and iterative methods.

Matrix Powers

If $T(x) = Ax$, then repeatedly applying T gives:

$$T^k(x) = A^k x$$

This is used in:

- **Dynamic systems**: Modeling population growth, material degradation, etc.
- **Iterative Solvers**: Successive approximations using power methods.

28.15 Linear Transformations and Differential Equations

In many physical systems, especially in civil engineering (e.g., vibrations of a bridge, thermal conduction in a beam), **systems of differential equations** arise, which can be written using linear transformations.

System of ODEs

$$\frac{dx}{dt} = Ax$$

Here, A is the matrix representing a linear transformation. The solution involves:

$$x(t) = e^{At} x(0)$$

Where e^{At} is the **matrix exponential**, which may be computed via diagonalization or Jordan forms.

Practical Examples

- Heat conduction modeled using Fourier's law (linear diffusion operator)
- Frame deflection using beam bending equations (linear elasticity)
- Modal vibration analysis (linear system with eigen-decomposition)

28.16 Transformations in Finite Element Methods (FEM)

In **Finite Element Analysis (FEA)**, coordinate transformations are used extensively:

Local to Global Coordinate Transformations

To assemble the global stiffness matrix, each local element matrix must be transformed:

$$K^{(global)} = T^T K^{(local)} T$$

Where T is the transformation matrix depending on element orientation.

Affine Transformations

Used to map:

- Reference elements (e.g., unit triangles) to physical elements in meshes.
- Jacobian matrices define these mappings, and their determinants indicate area or volume scaling.

28.17 Orthogonal Transformations

A transformation T is **orthogonal** if its matrix A satisfies:

$$A^T A = I \Rightarrow A^{-1} = A^T$$

Orthogonal transformations preserve:

- **Lengths**: $\| T(x) \| = \| x \|$
- **Angles**: $\langle T(x), T(y) \rangle = \langle x, y \rangle$

Examples:

- **Rotations** (no distortion, used in simulations)
- **Reflections** (used in symmetry analysis)

Relevance to Civil Engineering:

- Used in aligning axes in structural design
- Important in computer graphics for CAD software
- Ensure numerical stability in simulations (QR decomposition)