Discrete Mathematics
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Lecture -42
Counting Using Principle of Inclusion-Exclusion

So, hello everyone, welcome to this lecture the plan for this lecture is as follows. In this lecture we will introduce the principle of inclusion-exclusion and we will see some of its applications to various counting problems.

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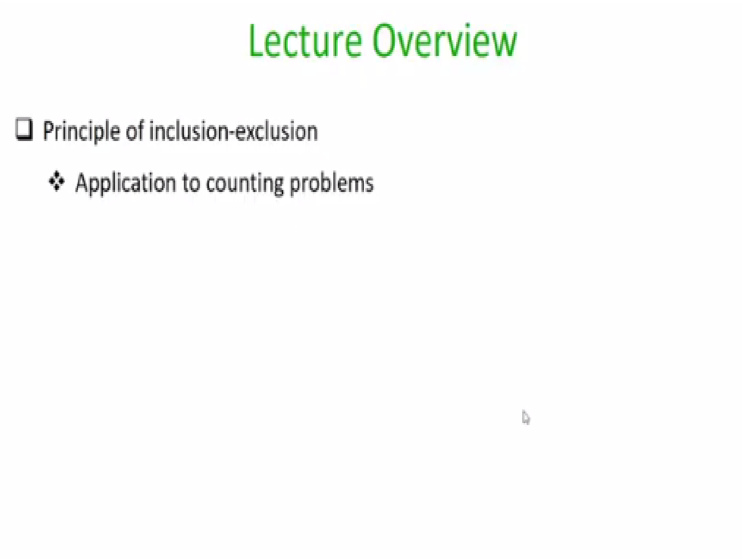

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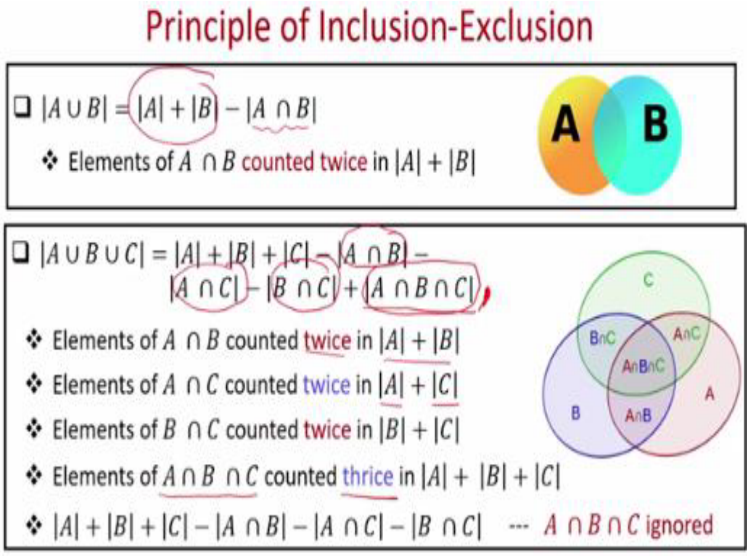

So, what exactly is the principle of inclusion-exclusion? Well it basically says that if you want to find out the cardinality of the union of 2 sets then it is same as taking the summation of cardinalities first of the individual sets and subtracting the cardinality of the intersection of the 2 sets. And why this is true because if we see pictorially or if we follow the Venn diagram method then if we just add up the cardinalities of the A set and B set then the common portion or the elements which are common to both sets A and B are counted twice.

So, to compensate or to avoid this over counting we basically subtract the cardinality of the intersection of the A set and the B set. Now extending this principle to the case of 3 sets if we want to find out the cardinality of the union of 3 sets then it is the summation of the cardinalities of the individual sets. Then we have to take the difference of the cardinalities of 2 sets at a time and then again we have to add the cardinality of the intersection of all the 3 sets.

Again we can prove it easily using the Venn diagram method. So, why we are subtracting the cardinality of $A \cap B$, $A \cap C$ and $B \cap C$. Because the elements in $A \cap B$ are counted twice if we add individually the cardinalities of the A set and B set. Similarly when we are adding up the cardinalities of A set and C set the common portion or the elements common to A and C are included twice and so on.

Whereas the elements which are common to all the 3 sets they are counted thrice. If we just add up the cardinality of A B and C and if we do not add this cardinality of intersection of A B and C set in this overall formula then because since we are subtracting out the common portion between A and B, common portion between A and C and common portion between B and C their cardinalities, if we do not do this extra addition then the contribution of the common elements in the 3 sets is ignored. So, that is why this plus is there. Now we have seen the principle of inclusion-exclusion for this case of 2 sets, for the case of 3 sets.

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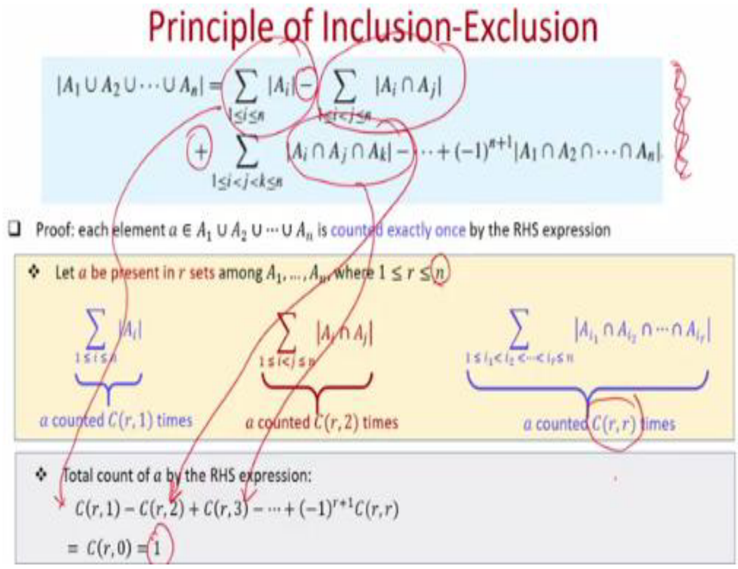

We can generalize it to the case of n sets. So, the general formula says the following. If you have n sets they may be disjoint, there might be overlaps and so on. Then the formula says that if you want to find out the cardinality of the union of n sets then this is this formula. That means it says that it is same as you have to first individually take the summation of the cardinalities of the individual sets.

Then you have to subtract the cardinality of pairwise intersection of sets. Then you have to add the cardinality of intersection of triplets of sets and so on. That means alternately, first we add then we subtract and we add and we subtract and so on. So, we have to prove that this formula is correct. You can use proof by induction but even without using proof by induction over n we can prove it.

So, the idea here will be the following, so consider an element a which is present in the union of the n sets. It might be present in just one of the sets, it might be present in 2 of the sets we do not know in how many sets the element is present. We have to show that if at all an element a is present in the union of n states then by this complex looking argument, complex looking formula in the right hand side expression the element a is counted exactly once.

And that is what we want to prove here basically. So, imagine that the element a is present in r number of sets out of the n sets where r is at least 1. Because we are considering the case where element is present in the union of all the n sets. So, if it is present in the union of all in n sets it is possible only when it is present in at least 1 of the n sets. And it might be the case that it is present in all the n sets. So, that is the value of r is in the range 1 to n.

Now what we have to show is our goal is to show that by the RHS expression the element A is counted exactly once overall. So, for that, we observe that the first part of the formula on the right hand side is where we are taking the summation of the cardinalities of the individual sets A_1 to A_n. Now because of this, the element a will be counted C(r, 1) number of times. So, for instance if the element a is present, say in the first r sets, then because when we are taking the cardinality of A_1 the element a was counted once. When we are taking the cardinality of A_2 the element a is again counted once. When we are taking the cardinality of A_3 the element a is again counted once and so on. When we are taking the cardinality of a_{r+1} we are not increasing the count of the element a. So, that means through the first part of the formula in my right hand side expression the element a is counted these many number of times : C(r,1).

Now let us take the second part of the formula where we are taking the cardinality of intersection of 2 sets at a time. So, basically it is like saying the following we are taking the cardinality of $A_1 \cap A_2$, $A_1 \cap A_3$ and like that $A_1 \cap A_n$ and then we are taking the cardinality of $A_2 \cap A_3$, $A_2 \cap A_4$ and so on by the second formula. So, we have to find out, and when we are taking the cardinalities of this pairwise intersection of sets, how many times the element a is getting counted. For the moment we are forgetting about this minus here, we are just trying to find out what will be the effect or how many times the element a will be counted when we are taking the cardinality of intersection of 2 sets at a time. So, as per our assumption the element is a is present in r number of sets. When we are taking the intersection of 2 sets at a time, element a will be counted only when both the sets where we are taking whose intersection we are taking element a is present.

If we are taking intersection of 2 sets where in one of the sets element is a is not present then the overall contribution for the cardinality of intersection of those 2 sets for the count of a will be 0. So, because of this overall, I can say that because of the second part of the expression the element a is counted C(r, 2) number of times. In the similar way, I can say that when we are taking the cardinality of intersection of 3 sets at a time, then element a will be counted only when the 3 sets whose intersection we are taking all of them has the small element a. If element a is not present in even in one of them then it will not be present in the intersection of 3 sets. Due to that because of considering the cardinality of intersection of 3 sets at a time the element a will be counted C(r, 3) number of times. And then if we continue this argument, we can say that when we are taking the cardinality of intersection of r sets at a time the element a will be counted exactly once namely C(r, r) number of times. And after that when we are taking the cardinality of intersection of r + 1 sets at a time, r + 2 sets at a time through those parts of my RHS formula the element a will not be counted at all.

Because as per our assumption the element is a is present only in r number of sets. So, now what we can say is that the total count of this element a by the RHS expression is the following C(r, 1) – C(r, 2) + C(r, 3) - … $(-1)^{r+1}$C(r, r). This is because through the first part of the expression the element A is counted C(r, 1) number of times, through the second part of my expression the count of element A gets decremented by C(r, 2) number of times.

Why decremented? Because of this minus. Then due to the third part of the expression, the count of element a gets incremented by C(r, 3) number of times and so on. And we know that this expression is nothing but C(r, 0). This follows from the properties of the combinatorics function or the binomial coefficients. So, it turns out that this summation is same as C(r, 0). And C(r, 0) is nothing, but 1.

So, that shows that by the expression in your right hand side the element a which is present in the union of n sets is counted exactly once. If there is an element which is not at all there in the union of n sets it will not be counted at all by the formula.

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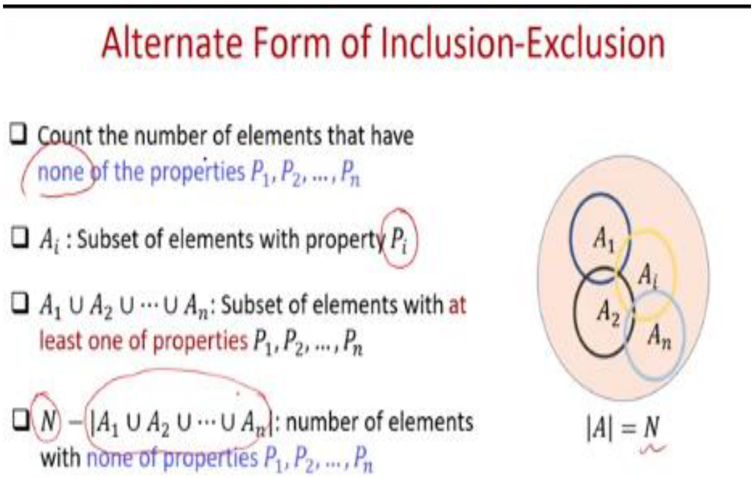

Now what we will do is, we will look into an alternate form of inclusion-exclusion which is very powerful and it is this alternate form of inclusion-exclusion which we use in varieties of problems. So, what happens in this alternate form is that we will be facing scenarios where we want to count the number of elements from a set A which has n number of elements and we will be interested to count the number of elements in this set which has none of the properties say P_1, P_2, P_n.

So, P_1, P_2, P_n will be some abstract properties and we will be interested to find out the number of elements in the set A which neither have the property P_1 and nor the property P_2 , nor the property P_3 and so on. So, for that what we will do is we will first identify the subset A_i consisting of elements which has property P_i. I stress A_i is the set of elements in the set A which has the property P_i.

And if I take the union of the subsets A_1 to A_n, I get all the elements in the set A which has either the property P_1 or the property P_2 or the property P_3 or so on. That means it will have at least one of the properties P_1 to P_n, but that is not what we want. We are interested in the elements which do not have any of these properties. So, it is easy to see that now the desired answer or the number of elements which do not have any of the properties P_1 to P_n will be the difference of the cardinality of A which is n and the unique cardinality of the union of n sets.

And now we will apply the rule of inclusion-exclusion to find out the cardinality of the union of the n sets because we know how to find that. So, this is the alternate form of inclusion-exclusion.

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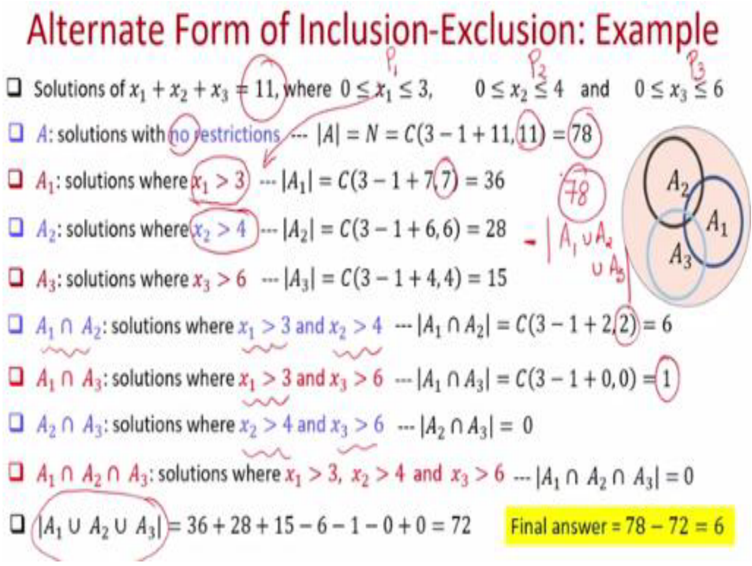

So, let me demonstrate this alternate form of inclusion-exclusion through some examples. Suppose we want to find out the number of solutions for this equation : $x_1 + x_2 + x_3 = 11$; number of integer solutions. And my restrictions are x_1 should be in the range 0 to 3, x_2 should be in the range 0 to 4 and x_3 should be in the range 0 to 6. Now using the; methods that we have seen till now we cannot find it directly.

But what we will do is the following : we will use the alternate form of inclusion-exclusion. So, we will first find out the cardinality of the universal set. Universal set in this context will be the set of all solutions where I do not put any restrictions on x_1, x_2, x_3. So, they could be anything, In fact I can have a solution where x_1 is 11 as well. And I know the cardinality of this universal set because that comes from the formula for combinatorics with repetition.

So, it is equivalent to saying that we have to select 11 bills and we are given bills of either denominations x_1, x_2 and x_3. And I have no restriction because that is what is the interpretation of the set A. Now I let me define a set A_1 to be the set of solutions for this equation where x_1 is greater than 3. So, now you see your property P_1 was that your solution should not have x_1 more than 3. But I am now finding solutions which violate the property P_1 namely solutions where x_1 is allowed to be more than 3.

In fact x_1 has to be more than 3 and we know how to find the cardinality of A_1 set. If x_1 is more than 3 that means I have to definitely pick 4 bills of x_1 type. So, I will be left with now selecting 7 more bills without any restriction and the number of ways of doing that is 36. In the same way my property P_2 is that my solution x_2 should be strictly less than equal to 4. But now I try to find out solutions which violate the property P_2 namely solutions where x_2 is definitely greater than 4.

And let A_2 be the set of such solutions the cardinality of A_2 will be this : 28. And similarly my property P_3 could be that I am interested in solutions where x_3 should be less than equal to 6. But then I try to find out solutions which violate this property and define the set of such solutions to be A_3 and the cardinality of the set A_3 will be this: 15. And now it is easy to see that my overall solution the number of overall solutions will be the difference of the cardinality of the universal set which is 78 and the union of, and the cardinality of the union of the set A_1, A_2 and A_3.

Now to find out the cardinality of the sets A_1, A_2, A_3, I have to use the principle of rule of inclusion-exclusion. For that I need the cardinality of a pairwise intersection of sets. So, let us find out cardinality of the set $A_1 \cap A_2$. So, $A_1 \cap A_2$ means the solutions where both x_1 is greater than 3 as well as x_2 is greater than 4. And it is easy to see that the number of solutions of such type will be this : 6.

Because if x_1 is greater than 3 that means x_1 is at least 4. That means I have picked 4 bills of type x_1 and if x_2 is greater than 4 that means x_2 is at least 5 that means I have picked 5 bills of type x_2. So, that means overall 9 bills have been already picked and I am now left with the problem of choosing 2 bills of either denominations x_1 x_2 x_3 without any restrictions. Similarly the cardinality of $A_1 \cap A_3$ will be this: 1.

This is because if x_1 is greater than 3 that mean x_1 is at least 4 and x_3 greater than 6 means x_3 is greater than equal to 7. That means I have already chosen 11 bills that means there is only one solution possible where x_1 is greater than 3 and x_3 is greater than 6. Namely we take x_1 to be 4 and x_3 to be 7. That is why the number of solutions or the cardinality of $A_1 \cap A_3$ is 1. Similarly if I take $A_2 \cap A_3$ it turns out that there are no solutions.

Because I cannot have any solutions where x_2 is greater than 4 and x_3 is greater than 6 because overall the sum of x_1, x_2, x_3 should be 11. If I take the intersection of A_1, A_2, A_3 again there are no solutions. And now I have all the things to find out the cardinality of union of A_1, A_2 A_3 and as I said if I subtract it from my universal set cardinality then that will give me the number of required solutions.

So, now you can see that how I use the alternate form of inclusion-exclusion for solving this problem.

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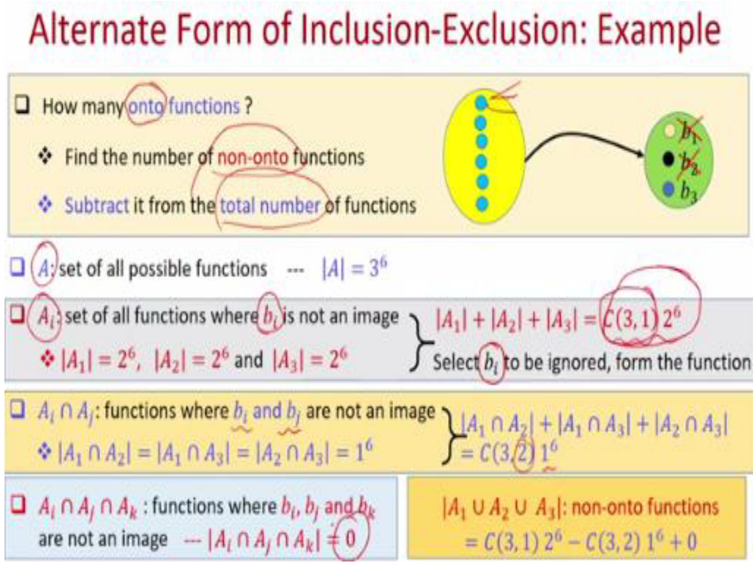

Let us use alternate form of inclusion-exclusion to solve another problem. So, you are given 2 sets a set A consisting of 6 elements and another set B consisting of 3 elements and I am interested to find out how many onto functions are there. We already have found the answer for this in one of our earlier exercises using Sterling numbers of second type. But now I will be using the alternate form of inclusion-exclusion to solve this problem.

So, for finding this and using the alternate form of inclusion-exclusion what we will do is, we will find out the total number of functions and from that we will subtract the total number of non-onto functions. So, let us define the set A to be the set of all functions and it is easy to see that the cardinality of the set A is 3^6. Because I have 6 elements; for each of the elements I have 3 possible images to pick from.

Mind it the set A has all the functions which are onto as well as all the functions which are non onto. Now I will be subtracting the number of non-onto functions. For that I define a subset A_i to be the set of all functions where the element b_i is not an image. So, it is like saying the following: An onto function will have the property that b_1 has a pre image b_2 has an pre image as well as b_3 has a pre image.

So, these are my 3 properties. If any of these 3 properties is violated I get a non-onto functions. So, that is what I am now trying to find out how many ways I can violate this properties P_1, P_2, P_3. So, I try to analyze how many ways I can violate the property P_i, so property P_i is the element b_i is an image whenever I am picking a function. So, violation of that property will be that element b_i is not chosen as an image when a function is from the A set to the B set.

And let A_i denote the set of all such functions where the element b_i do not have any pre image. So, for instance the cardinality of the set A_1 will be 2^6 because if b_1 is not allowed to be a possible image then for each of the 6 elements I have 2 possible images to assign: either b_2 or b_3. In the same way the cardinality of A_2 also will be 2^6 because A_2 means the element b_2 is not allowed to be a possible image.

That means the only images could be b_1 or b_3 and so on. So, I can say that if I take the cardinality; if I add the cardinalities of A_1, A_2, A_3 then that is same as the following. You first select the element b_i which should be ignored that means which should not have a pre image. And then form a function from the A set to the B set with respect to the remaining 2 elements.

So, the element b_i which has to be ignored it can be chosen in C(3, 1) ways and now you are left with only 2 elements.

So, each of the elements in the A set can be assigned those 2 images. So, this will be the overall count if I take the summation of the cardinalities of the sets A_1, A_2, A_3. Now if I take the pairwise intersections of A_i and A_j sets, then basically they denote the functions where neither the element b_i nor the element b_j can be the possible images. So, for instance the cardinality of the set $A_1 \cap A_2$ will be 1^6.

Because $A_1 \cap A_2$; means the following. You cannot have element b_1 as a possible image, you cannot have the element b_2 as a possible image. That means we are considering the function where all the elements are mapped to b_3. So, there is only 1^6 such functions basically 1. Same way the cardinality of $A_1 \cap A_3$ will be 1^6 and so on. So, again we can say that if we take the summation of the cardinalities of the pairwise intersection of sets then that will be same as the following.

You find out the 2 elements which have to be ignored that means which are not allowed to be the possible images. So, you can do that in C(3, 2) ways and then you will be left with only one element which is the only allowed image. And then if I take the cardinality of intersection of 3 sets at a time then the interpretation of that will be: I am considering functions where neither the element b_i, b_j and b_k are allowed to be possible images.

In this example since I have only 3 possible images to choose from, I am saying that I want to design a function where none of those 3 elements can be the images. And I cannot have any such function, so that is why the cardinality of the intersection of triplet subset will be 0 in this case. So, as I said now the union of A_1, A_2, A_3 will give you all the non-onto functions. Because the union of these 3 sets will have those functions where either b_1 is not allowed to be an image or b_2 is not allowed to be an image or b_3 is not allowed to be an image.

And as per the rule of inclusion-exclusion the number of non-onto functions will be this and if you subtract this quantity from this value the size of the universal set that will give you the number of onto functions.

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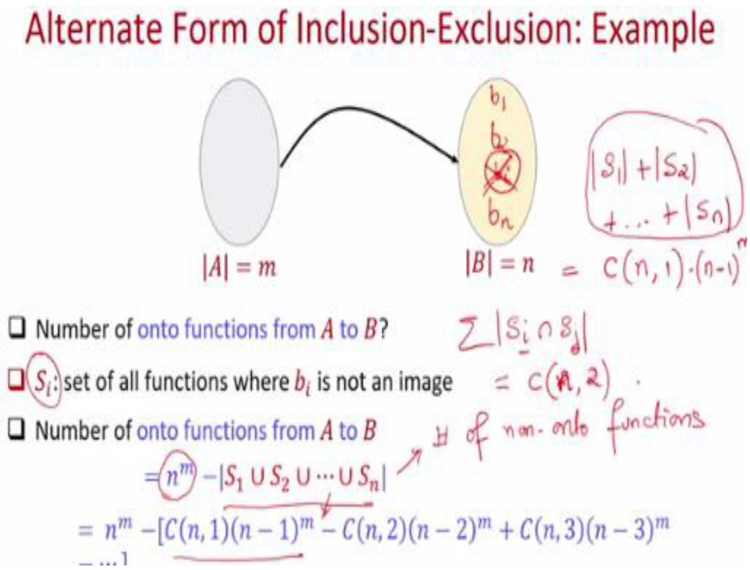

So, now let us generalize this formula, so in the previous case we had the case where the first set has 6 elements and the set of possible images has 3 elements, namely my m was 6 and my n was 3. So, now we want to generalize it, we have a set A having m elements and a set B having n elements we want to find out the number of onto functions. So, for that I define a set S_i to be the set of all functions where the element b_i is not allowed to be an image.

So, I am assuming that my elements of the set B are b_1 to b_n and I am trying to find out; I am defining the set S_i to be the set of all functions where all the elements in the set B are allowed to be images except the element b_i. And as we did in the demonstration in the earlier slide the total number of onto functions will be the difference of the number of all possible functions which include both onto and non-onto functions and the number of non onto functions.

And the number of non-onto functions is given by the cardinality of union of the sets S_1 to S_n. So, this is the number of non-onto functions. And if we expand this cardinality of the union of S_1 to S_n then it will turn out to be this. So, the first term here denotes basically that the cardinality of S_1

+ S_2 +…+ S_n is nothing but you decide the single element which is not allowed to be image that can be done in C(n, 1) ways.

And then the remaining n – 1 elements in the set B could be the images for the elements in the set A. So overall contribution of the cardinality of S_1 to S_n will be this. Similarly if I want to take the pairwise intersection and their cardinalities and sum them up then it is equivalent to saying that $S_i \cap S_j$ basically denotes all those functions where neither b_i nor b_j can be the images.

That means the remaining n - 2 elements can be the images; now this i and j can be; any i and j from the set 1 to n the indexes. So, that is why it is equivalent to saying that you decide, you choose the 2 elements i and j or b_i or b_j which are not allowed to be the images and then your function will have the remaining n - 2 elements as the possible images and so on.

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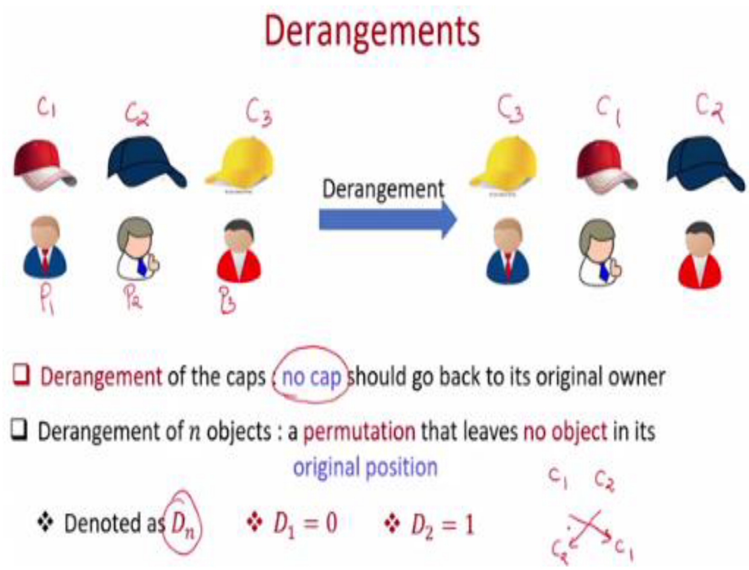

Now our last case study for the alternate form of inclusion-exclusion is the number of derangements. So, what exactly is a derangement, so imagine you have 3 persons, person 1 (P_1), person 2 (P_2) and person 3 (P_3) and they have their respective caps, cap 1 (C_1). cap 2 (C_2) and cap 3 (C_3). Now the derangement of caps is an arrangement of the 3 caps, so that no cap go back to its original owner.

So, this is one of the derangements. Now P_1 gets the cap number C_3 P_2 gets cap number C_1 and P_3 gets cap number C_2. So, none of the person get backs its original cap. So, that is in the derangement and the derangement of n objects basically denotes a permutation of those objects such that it leaves no object in its original position.

And the number of derangements of n objects is denoted by this quantity D_n. So, D_1 is 0 because if there is only one object and you cannot derange it. It will be at its position, D_2 is one because if you have cap number 1 at position 1 and cap number 2 at position 1 and the only way to derange it is take cap number 1 to position 2 and take cap number 2 to position 1 and so on.

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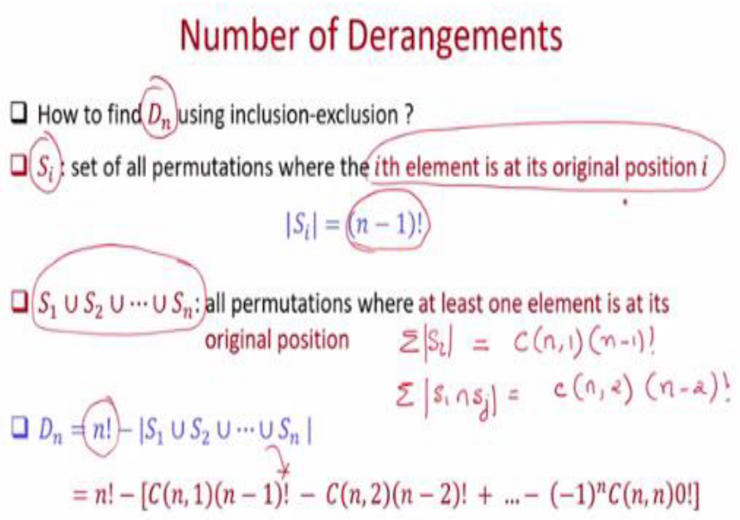

So, we want to find out the number of derangements namely the value of D_n. And there are several ways of doing that in the tutorial we will derive a recurrence relation. In this lecture we will see how to find out the value of D_n using the alternate form of inclusion-exclusion. So, again remember in the alternate form of the inclusion-exclusion you have to identify the property P_i and then you have to see in how many ways you can violate the property P_i.

So, property P_i with respect to the derangement is that you do not want the ith element to be at the position i. Because if ith element is still there at position i; then that is not a derangement. We want to violate that property, so let S_i denote the set of all possible permutations of n elements where the ith element is still at its position namely it is still at the position i. So, it is easy to see that the cardinality of the set S_i is $(n – 1)!$, because think of it as follows.

You have position 1, position i, position n you are finding all possible permutations where the element i is definitely at its position namely i it is still at position i. I do not know what is the status of the remaining n - 1 values. They may be at their position some of them may be at their position, none of them may be there at their position and so on. So, it is equivalent to saying that I am interested to permute the remaining n - 1 elements which can be done in $(n – 1)!$ ways.

So, that is the cardinality of the set S_i and as per our definition if I take the union of the sets S_1 to S_n it gives me all the permutations where at least one of the elements is still at its original position. It could be either the first element or the second element or the third element or the nth element, that means the arrangements or the permutations in the union of the sets S_1 to S_n are not derangements.

Because as per derangements; none of the n elements should be at its position. So, that means as per the alternate form of inclusion-exclusion the value of D_n will be the cardinality of the universal set. The universal set in this case will be the number of all possible arrangements or permutations of n elements. They could be derangement or they need not be derangements. So, this is n! and from that I have to subtract the cardinality of the union of the n sets.

Now, if I expand the cardinality of the union of n sets this will turn out to be this. So, if I take the effect of the summations of all S_i 's the cardinality of the summations of S_i this will be same as you pick the element i which is still at its position namely at position i. That could be done in C(n, 1) ways and then the cardinality of each set S_i is $(n – 1)!$. Similarly if I take the cardinality of intersection of S_i and S_j then the elements i and j can be chosen in C(n, 2) ways.

That means $S_i \cap S_j$ denotes all those permutations where the ith element is at position i and the jth element is still at position j. That means the remaining n - 2 elements can be permuted in any order and so on. So, that will give you the value of $D^n = n! - [C(n,1)(n-1)! - C(n,2)(n-2)! + \ldots + (-1)^n C(n,n)0!]$.

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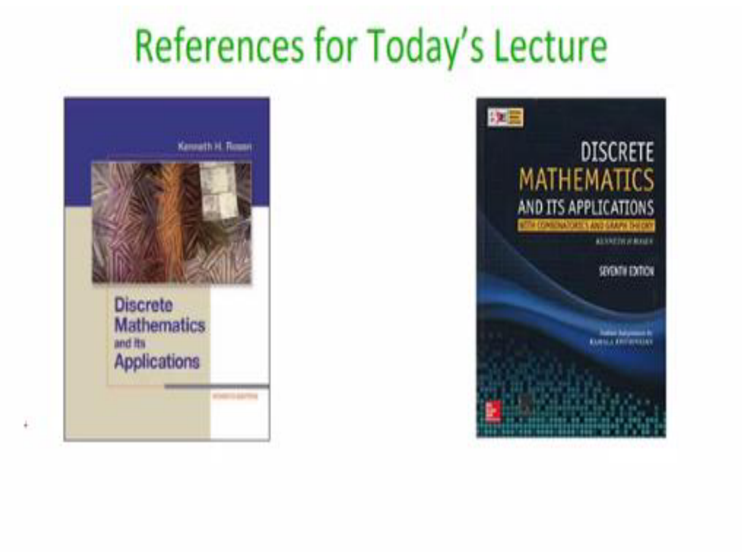

So, these are the references for today's lecture and with that I end this lecture. Just to summarize in this lecture we discussed the principle of inclusion-exclusion, we derived the formula for that and we saw an alternate form of inclusion-exclusion and some case studies for the alternate form of inclusion and exclusion, thank you.