Chapter 11: Fourier Transform and Properties

Introduction

The **Fourier Transform** is one of the most powerful mathematical tools used in engineering and applied sciences. It allows us to analyze and represent functions in the frequency domain rather than the time or spatial domain. In civil engineering, Fourier transforms find applications in areas like structural analysis, signal processing in sensor data, vibration analysis, groundwater modeling, and heat transfer problems.

Understanding the fundamental definitions, computation techniques, and properties of the Fourier Transform is essential for students and professionals who need to analyze periodic and non-periodic phenomena in the frequency domain.

11.1 Definition of Fourier Transform

Let $f(t)$ be a function defined on the entire real line, i.e., $t \in (-\infty, \infty)$. The **Fourier Transform** of $f(t)$ is a complex-valued function $F(\omega)$ defined by:

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

where:

- ω is the angular frequency in radians per second,
- i is the imaginary unit ($\sqrt{-1}$).

This transform converts the time-domain signal $f(t)$ into its frequency-domain representation $F(\omega)$.

Inverse Fourier Transform

To recover $f(t)$ from its Fourier Transform $F(\omega)$, we use the **Inverse Fourier Transform**:

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

This two-way transformation is central to signal processing, vibrations, and analysis of heat conduction in civil engineering systems.

11.2 Conditions for Existence (Dirichlet's Conditions)

A function $f(t)$ will have a Fourier Transform if:

1. $f(t)$ is absolutely integrable over $(-\infty, \infty)$, i.e.,

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$

2. $f(t)$ has a finite number of discontinuities in any finite interval.
3. $f(t)$ has a finite number of maxima and minima in any finite interval.

These are sufficient but not necessary conditions.

11.3 Properties of the Fourier Transform

Understanding the properties helps in simplifying the process of taking Fourier transforms of complex functions.

1. Linearity

$$\mathcal{F}\{af(t) + bg(t)\} = aF(\omega) + bG(\omega)$$

Linearity allows superposition of transformations.

2. Time Shifting

$$\mathcal{F}\{f(t - t_0)\} = e^{-i\omega t_0} F(\omega)$$

Shifting a function in time corresponds to a phase shift in frequency.

3. Frequency Shifting

$$\mathcal{F}\{e^{i\omega_0 t} f(t)\} = F(\omega - \omega_0)$$

Multiplication by a complex exponential in time domain shifts the frequency spectrum.

4. Time Scaling

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

Scaling the time domain compresses or stretches the frequency domain.

5. Differentiation in Time Domain

$$\mathcal{F}\left\{\frac{d^n f(t)}{dt^n}\right\} = (i\omega)^n F(\omega)$$

This is particularly useful in solving differential equations.

6. Convolution Theorem

If $f(t) * g(t)$ is the convolution of f and g, then:

$$\mathcal{F}\{f * g\} = F(\omega) \cdot G(\omega)$$

This transforms convolution in the time domain to multiplication in the frequency domain.

7. Parseval's Theorem

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

It preserves the energy of a signal across time and frequency domains.

11.4 Fourier Cosine and Sine Transforms

When the function is defined only on $[0, \infty)$, we often use **Fourier Cosine Transform (FCT)** and **Fourier Sine Transform (FST)**.

Fourier Cosine Transform

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos(\omega t) dt$$

Inverse Fourier Cosine Transform

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\omega) \cos(\omega t) d\omega$$

Fourier Sine Transform

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin(\omega t) dt$$

Inverse Fourier Sine Transform

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\omega) \sin(\omega t) d\omega$$

These transforms are commonly applied in solving boundary value problems in civil engineering (e.g., heat flow through semi-infinite slabs, vibration of beams).

11.5 Applications in Civil Engineering

- **Vibration Analysis**: Used in modal analysis of buildings and bridges to evaluate natural frequencies and responses to dynamic loads.
- **Heat Transfer**: Solves transient heat conduction problems in solids.
- **Groundwater Flow**: Analyzing dispersion of pollutants in aquifers.
- **Signal Processing**: Filtering and analyzing data from sensors and structural health monitoring systems.
- **Seismic Analysis**: Interpreting frequency content of earthquake ground motions.

11.6 Important Standard Fourier Transforms

Function $f(t)$	Fourier Transform $F(\omega)$		
1	$2\pi\delta(\omega)$		
$\delta(t)$	1		
(e^{-a	t	},\ a > 0)	$\frac{2a}{a^2+\omega^2}$
$e^{-at}u(t),\ a > 0$	$\frac{1}{a+i\omega}$		
$\text{rect}(t/T)$	$T \cdot \text{sinc}(\omega T/2)$		
$\cos(\omega_0 t)$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$		
$\sin(\omega_0 t)$	$\frac{\pi}{i}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$		

11.7 Derivation of Fourier Transform of Common Functions

Let's derive a few Fourier Transforms of frequently used functions in civil engineering applications.

11.7.1 Fourier Transform of Rectangular Pulse

Let

$$f(t) = \begin{cases} 1 & |t| \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

This is a rectangular pulse of width T.

Fourier Transform:

$$F(\omega) = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\omega t} dt = \left[\frac{e^{-i\omega t}}{-i\omega} \right]_{-T/2}^{T/2}$$

$$F(\omega) = \frac{2\sin(\omega T/2)}{\omega} = T \cdot \text{sinc}\left(\frac{\omega T}{2\pi} \right)$$

where $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$

This transform is fundamental in signal analysis.

11.7.2 Fourier Transform of Exponential Decay

Let $f(t) = e^{-at}u(t)$, where $a > 0$, $u(t)$ is the unit step function.

$$F(\omega) = \int_0^{\infty} e^{-at} e^{-i\omega t} dt = \int_0^{\infty} e^{-(a+i\omega)t} dt$$

$$F(\omega) = \left[\frac{e^{-(a+i\omega)t}}{-(a+i\omega)} \right]_0^{\infty} = \frac{1}{a+i\omega}$$

11.8 Example Problems

Example 1: *Find the Fourier Transform of* $f(t) = e^{-2t}u(t)$

$$F(\omega) = \int_0^{\infty} e^{-2t} e^{-i\omega t} dt = \int_0^{\infty} e^{-(2+i\omega)t} dt = \frac{1}{2+i\omega}$$

Example 2: ***Find the Fourier Sine Transform of*** $f(t) = e^{-at}$***,*** $a > 0$

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at} \sin(\omega t) dt$$

This is a standard integral:

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \cdot \frac{\omega}{a^2 + \omega^2}$$

Example 3: ***Use the time-scaling property to find the Fourier Transform of*** $f(t) = \mathbf{rect}(2t)$

We know:

$$\mathcal{F}\{\text{rect}(t)\} = \text{sinc}\left(\frac{\omega}{2}\right)$$

Using time scaling: $f(at) \Rightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$

$$\mathcal{F}\{\text{rect}(2t)\} = \frac{1}{2} \cdot \text{sinc}\left(\frac{\omega}{4}\right)$$

11.9 Solving Differential Equations Using Fourier Transform

Let us consider a second-order linear ODE:

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = f(t)$$

Taking the Fourier Transform of both sides:

$$\mathcal{F}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathcal{F}\left\{\frac{dy}{dt}\right\} + 2\mathcal{F}\{y\} = \mathcal{F}\{f(t)\}$$

$$\Rightarrow (-\omega^2 Y(\omega)) - 3(i\omega Y(\omega)) + 2Y(\omega) = F(\omega)$$

$$Y(\omega)\left[-\omega^2 - 3i\omega + 2\right] = F(\omega) \Rightarrow Y(\omega) = \frac{F(\omega)}{-\omega^2 - 3i\omega + 2}$$

Take the inverse transform of $Y(\omega)$ to get $y(t)$.

This approach is crucial for solving boundary value problems, especially where Green's functions are hard to construct.

11.10 Fourier Transform in Two Dimensions (2D Fourier Transform)

For a function $f(x, y)$, its 2D Fourier Transform is:

$$F(u, v) = \iint_{-\infty}^{\infty} f(x, y) e^{-i(ux+vy)} dx\, dy$$

Inverse 2D Fourier Transform:

$$f(x, y) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} F(u, v) e^{i(ux+vy)} du\, dv$$

Applications in Civil Engineering:

- Used in **image processing** (e.g., in satellite or drone imaging of terrain or structures).
- Applied in **soil mechanics** and **wave propagation** studies.

11.11 Discrete Fourier Transform (DFT)

In real-world applications (such as sensor data in structural health monitoring), signals are sampled. The **Discrete Fourier Transform (DFT)** is used to analyze such sampled signals.

For a finite sequence $f_0, f_1, \dots, f_{N-1}$, the DFT is:

$$F_k = \sum_{n=0}^{N-1} f_n e^{-2\pi i k n / N}, \quad k = 0, 1, \dots, N-1$$

Inverse DFT:

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{2\pi i k n / N}$$

This is implemented efficiently using the **Fast Fourier Transform (FFT)** algorithm.

11.12 Use of FFT in Civil Engineering Applications

- **Real-time Vibration Monitoring** of buildings and bridges using accelerometers.
- **Dynamic Soil Testing** using wave propagation.
- **Modal Analysis** for resonance detection in concrete structures.
- **Frequency analysis of wind loads** on tall buildings.
