Chapter 23: Linear Independence

Introduction

In the study of vector spaces and systems of linear equations, the concept of **linear independence** plays a foundational role. It helps determine whether a set of vectors contains redundancy and whether they span a unique subspace. In civil engineering, linear independence is used in structural analysis, stability of structures, finite element methods, and solving systems of equations for unknowns in mechanical systems.

This chapter develops the concept of linear independence in the context of vector spaces, providing the definitions, theorems, examples, and necessary techniques to verify and work with linearly independent sets.

23.1 Vector Spaces and Basis (Recap)

Before we discuss linear independence, recall the basic terms:

- A **vector space** is a set of vectors closed under vector addition and scalar multiplication.
- A **basis** of a vector space is a linearly independent set of vectors that spans the entire space.
- The **dimension** of a vector space is the number of vectors in a basis.

23.2 Linear Combination of Vectors

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be vectors in a vector space V. A **linear combination** of these vectors is any vector of the form:

$$\vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_n \vec{v}_n$$

where $a_1, a_2, \ldots, a_n \in R$ (or any field).

The idea of linear independence revolves around whether the **only** solution to this combination being the zero vector is the **trivial solution**.

23.3 Definition of Linear Independence

A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in a vector space V is said to be **linearly independent** if:

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_n \vec{v}_n = \vec{0}$$

implies that $a_1 = a_2 = \cdots = a_n = 0$. Otherwise, the set is called **linearly dependent**.

Key idea: If at least one vector in the set can be written as a linear combination of the others, the set is linearly dependent.

23.4 Geometric Interpretation

In R^2:

- Two vectors are **linearly independent** if they are **not collinear**.
- Geometrically, they span a plane.

In R^3:

- Three vectors are linearly independent if they are **not coplanar**.
- They span the entire three-dimensional space.

23.5 Algebraic Criterion for Linear Independence

To test whether a set of vectors is linearly independent:

1. Form a linear combination set equal to the zero vector.
2. Convert it to a homogeneous system of equations.
3. Solve the system:
   - If the only solution is the trivial solution (all coefficients = 0), the vectors are linearly independent.
   - If there exists a non-trivial solution, the set is linearly dependent.

23.6 Matrix Approach: Row Reduction

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be vectors in R^m, and form the matrix:

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$$

Steps:

- Perform **Gaussian elimination** to reduce A to its **row echelon form** (REF).
- Count the number of **pivot columns**:
  - If number of pivot columns = number of vectors → linearly independent.
  - If number of pivot columns < number of vectors → linearly dependent.

23.7 Examples

Example 1

Check if the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

are linearly independent.

Solution: Form the matrix:

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Row reduce:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank = 2 < 3 → **Linearly dependent**

Example 2

Determine if

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent.

This is the standard basis of R^3, and clearly linearly independent.

23.8 Properties of Linearly Independent Sets

1. A subset of a linearly independent set is also linearly independent.
2. If a set spans V and is linearly independent, it is a **basis**.
3. Any set containing the zero vector is linearly dependent.
4. In R^n, any set of more than n vectors is linearly dependent.

23.9 Applications in Civil Engineering

- **Structural Analysis**: Forces acting at joints in trusses must be linearly independent to determine unique solutions.
- **Finite Element Method**: Shape functions must form a linearly independent basis to represent solutions uniquely.
- **Equilibrium Conditions**: Vector representation of force and moment equilibrium uses linearly independent equations.
- **Material Behavior**: Stress-strain relationships are often expressed using linearly independent vector systems.

23.10 Exercises

1. Determine whether the following vectors are linearly independent:

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

2. Show that the set $\{(1,2),(2,4)\}$ is linearly dependent.

3. For what values of a is the set $\{(1,2),(a,1)\}$ linearly independent?

4. In R^4, is it possible for 5 vectors to be linearly independent? Justify your answer.

5. Using Gaussian elimination, check the independence of:

$$\vec{v}_1=(1,2,3,4), \vec{v}_2=(0,1,1,0), \vec{v}_3=(1,3,4,4)$$

23.10 The Rank of a Matrix and Linear Independence

The **rank** of a matrix is the maximum number of linearly independent row (or column) vectors in the matrix. It plays a direct role in determining whether a system of vectors (or equations) is linearly independent.

Let A be a matrix with column vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Then:

- The column vectors of A are linearly independent **if and only if** $\text{rank}(A)=n$.

This provides a very efficient test: use Gaussian elimination or row reduction to determine the rank.

> **Engineering Insight:** In the analysis of structural systems like trusses or beams, stiffness and flexibility matrices must have **full rank** to ensure the system is determinate and stable.

23.11 The Wronskian and Linear Independence of Functions

In the context of **differential equations**, linear independence of functions is tested using the **Wronskian determinant**.

Let $f_1(x), f_2(x), \ldots, f_n(x)$ be functions differentiable up to order $n-1$. The **Wronskian** is defined as:

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

- If $W \neq 0$ for some x, the functions are linearly independent on that interval.

> **Engineering Application:** In civil engineering problems involving beam deflection, soil mechanics, or vibration, linear independence of solutions ensures a unique and physically valid deformation profile.

23.12 Generalization to Function Spaces

The concept of linear independence is not limited to finite-dimensional spaces like R^n. It applies to infinite-dimensional **function spaces**, such as spaces of continuous or differentiable functions.

Example:

The functions $1, x, x^2, x^3, \ldots$ are linearly independent in the space of all polynomials.

This leads to important applications in:

- **Fourier series**: Functions like $\sin(nx), \cos(nx)$ are orthogonal and linearly independent.
- **Structural dynamics**: Mode shapes in vibration analysis are linearly independent functions.

23.13 Linearly Dependent Systems in Engineering Practice

While linear independence is desirable in theory, **in real-world systems**, dependencies may arise due to:

1. **Redundancy in supports or members** (e.g., an extra bracing member).
2. **Measurement or computation errors**, causing near-linear dependence.
3. **Ill-conditioned matrices** in FEM or analysis software.

Example:

In a finite element mesh, if nodes are duplicated or too close together, stiffness matrices may become rank-deficient, indicating **dependency**.

> Engineers must either modify the system (remove redundancy) or use **regularization** or **numerical conditioning techniques**.

23.14 Maximal Linearly Independent Sets

Given a linearly dependent set of vectors, a **maximal linearly independent subset** can be extracted using row operations.

Algorithm:

1. Write vectors as columns of a matrix.

2. Perform row reduction to echelon form.
3. Identify pivot columns — the corresponding vectors form the maximal independent set.

This concept is important in:

- **Optimization problems**: Basis reduction in simplex method.
- **Sensor networks**: Removing redundant sensors whose data can be predicted by others.

23.15 Orthogonality and Linear Independence

Theorem:

If a set of non-zero vectors is **mutually orthogonal**, then it is **linearly independent**.

This is an especially important property in numerical methods.

Applications in Civil Engineering:

- **Modal analysis**: Mode shapes are orthogonal.
- **Least squares method**: Orthogonal basis simplifies calculations.

Proof Idea: Assume $\vec{v}_1, \dots, \vec{v}_n$ are orthogonal. Consider the equation:

$$a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n = \vec{0}$$

Take dot product with $\vec{v}_k$, and use orthogonality to show all $a_k = 0$.

23.16 Real-Life Structural Scenarios

1. **Overdetermined Structures**:

- Too many supports or members → linearly dependent force/moment equations.
- May lead to indeterminacy unless compatibility conditions are used.

2. **Bridge Design**:

- Independent equations of equilibrium are essential for analyzing forces in each beam or cable.

3. **Material Testing**:

- Stress-strain data is fitted to models with independent basis functions.

23.17 Summary Table: Quick Tests for Linear Independence

Scenario	Test Method	Result
Vectors in R^n	Row reduce matrix of vectors	Full rank → Independent
Functions	Wronskian $\neq 0$	Linearly independent
Orthogonal Vectors	Check dot products	If all 0 (except with self), then independent
Structural Equations	Count unique equilibrium equations	≤ DOF → Possibly independent