Chapter 31

Case Study II: GEORGE WASHINGTON BRIDGE

31.1 Theory

1 Whereas the forces in a cable can be determined from statics alone, its configuration must be derived from its deformation. Let us consider a cable with distributed load $p(x)$ **per unit horizontal projection** of the cable length (thus neglecting the weight of the cable). An infinitesimal portion of that cable can be assumed to be a straight line, Fig. 31.1 and in the absence of any horizontal load we have H =constant. Summation of the vertical forces yields

$$(+\downarrow)\,\Sigma F_y = 0 \Rightarrow -V + wdx + (V + dV) = 0 \quad (31.1\text{-a})$$
$$dV + wdx = 0 \quad (31.1\text{-b})$$

where V is the vertical component of the cable tension at x (Note that if the cable was subjected to its own weight then we would have wds instead of wdx). Because the cable must be tangent to T, we have

$$\tan\theta = \frac{V}{H} \quad (31.2)$$

Substituting into Eq. 31.1-b yields

$$d(H\tan\theta) + wdx = 0 \Rightarrow -\frac{d}{dx}(H\tan\theta) = w \quad (31.3)$$

2 But H is constant (no horizontal load is applied), thus, this last equation can be rewritten as

$$-H\frac{d}{dx}(\tan\theta) = w \quad (31.4)$$

3 Written in terms of the vertical displacement v, $\tan\theta = \frac{dv}{dx}$ which when substituted in Eq. 31.4 yields the **governing equation for cables**

$$-Hv'' = w \quad (31.5)$$

4 For a cable subjected to a uniform load w, we can determine its shape by double integration of Eq. 31.5

$$-Hv' = wx + C_1 \quad (31.6\text{-a})$$

$$-Hv = \frac{wx^2}{2} + C_1 x + C_2 \tag{31.6-b}$$

and the constants of integrations C_1 and C_2 can be obtained from the boundary conditions: $v = 0$ at $x = 0$ and at $x = L \Rightarrow C_2 = 0$ and $C_1 = -\frac{wL}{2}$. Thus

$$v = \frac{w}{2H} x(L - x) \tag{31.7}$$

This equation gives the shape $v(x)$ in terms of the horizontal force H,

5 Since the maximum sag h occurs at midspan ($x = \frac{L}{2}$) we can solve for the horizontal force

$$\boxed{H = \frac{wL^2}{8h}} \tag{31.8}$$

we note the analogy with the maximum moment in a simply supported uniformly loaded beam $M = Hh = \frac{wL^2}{8}$. Furthermore, this relation clearly shows that the horizontal force is inversely proportional to the sag h, as $h \searrow H \nearrow$. Finally, we can rewrite this equation as

$$r \stackrel{\text{def}}{=} \frac{h}{L} \tag{31.9-a}$$

$$\frac{wL}{H} = 8r \tag{31.9-b}$$

6 Eliminating H from Eq. 31.7 and 31.8 we obtain

$$\boxed{v = 4h\left(-\frac{x^2}{L^2} + \frac{x}{L}\right)} \tag{31.10}$$

Thus the cable assumes a parabolic shape (as the moment diagram of the applied load).

7 Whereas the horizontal force H is constant throughout the cable, the tension T is not. The maximum tension occurs at the support where the vertical component is equal to $V = \frac{wL}{2}$ and the horizontal one to H, thus

$$T_{\max} = \sqrt{V^2 + H^2} = \sqrt{\left(\frac{wL}{2}\right)^2 + H^2} = H\sqrt{1 + \left(\frac{wL/2}{H}\right)^2} \tag{31.11}$$

Combining this with Eq. 31.8 we obtain[1].

$$\boxed{T_{\max} = H\sqrt{1 + 16r^2} \approx H(1 + 8r^2)} \tag{31.12}$$

8 Had we assumed a uniform load w **per length of cable** (rather than horizontal projection), the equation would have been one of a catenary[2].

$$\boxed{v = \frac{H}{w}\cosh\left[\frac{w}{H}\left(\frac{L}{2} - x\right)\right] + h} \tag{31.13}$$

The cable between transmission towers is a good example of a catenary.

[1] Recalling that $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdot$ or $(1+b)^n = 1 + nb + \frac{n(n-1)b^2}{2!} + \frac{n(n-1)(n-2)b^3}{3!} + \cdots$; Thus for $b^2 << 1$, $\sqrt{1+b} = (1+b)^{\frac{1}{2}} \approx 1 + \frac{b}{2}$

[2] Derivation of this equation is beyond the scope of this course.