Chapter 31: Similarity of Matrices

Introduction

In linear algebra and its applications to civil engineering, the concept of *similarity of matrices* plays a fundamental role in simplifying complex matrix operations, especially in solving systems of equations, understanding stability in structural analysis, and reducing matrices to simpler forms such as diagonal or Jordan forms.

Matrix similarity captures the idea that two matrices represent the same linear transformation under different bases. This idea leads to important computational simplifications and helps in analyzing the qualitative properties of a system.

31.1 Definition of Similar Matrices

Let A and B be two square matrices of order n. We say that **matrix A is similar to matrix B** if there exists a nonsingular (invertible) matrix P such that:

$$B = P^{-1}AP$$

This relation is known as **matrix similarity**.

- $A \sim B$: Denotes that A is similar to B.
- The matrix P is called the **change-of-basis matrix**.

Properties:

1. **Reflexivity**: Every matrix is similar to itself. $A = I^{-1}AI \Rightarrow A \sim A$
2. **Symmetry**: If $A \sim B$, then $B \sim A$. Since $B = P^{-1}AP \Rightarrow A = PBP^{-1}$
3. **Transitivity**: If $A \sim B$ and $B \sim C$, then $A \sim C$. If $B = P^{-1}AP$ and $C = Q^{-1}BQ$, then $C = (QP)^{-1}A(QP) \Rightarrow A \sim C$

These properties show that matrix similarity is an **equivalence relation**.

31.2 Geometrical Interpretation

Matrix similarity represents the same linear transformation under two different coordinate systems (or bases).

In geometric terms:

- A linear transformation $T : V \to V$, represented by matrix A in basis β, can be represented by matrix B in another basis γ.
- Then $A \sim B$, and $B = P^{-1}AP$ where P transforms vectors from basis γ to β.

This is crucial in civil engineering where transformations of stress, strain, or displacement tensors under coordinate change occur frequently.

31.3 Invariant Properties under Similarity

If $A \sim B$, then A and B share several **invariant properties**:

1. Determinant:

$$\det(A) = \det(B)$$

2. Trace:

$$\mathrm{Tr}(A) = \mathrm{Tr}(B)$$

3. Rank:

$$\mathrm{Rank}(A) = \mathrm{Rank}(B)$$

4. Characteristic Polynomial:

$$\chi_A(\lambda) = \chi_B(\lambda)$$

5. Eigenvalues:

- Similar matrices have the **same set of eigenvalues** (including algebraic multiplicities).

These invariants are fundamental in analyzing system behavior such as **vibrations in structures**, **modal analysis**, and **stability analysis** in civil engineering.

31.4 Diagonalization and Similarity

A matrix A is said to be **diagonalizable** if it is similar to a diagonal matrix D:

$$D = P^{-1}AP$$

Where $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, the eigenvalues of A, and P is the matrix whose columns are the **linearly independent eigenvectors** of A.

Conditions for Diagonalizability:

- Matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.
- This is always true if:
  - A has n **distinct eigenvalues**, or
  - A is **symmetric** (especially relevant in civil engineering applications).

Diagonalization simplifies computation, especially when raising matrices to powers, as:

$$A^k = PD^kP^{-1}$$

This has implications in dynamic system simulations and solving systems of differential equations arising in civil structures.

31.5 Canonical Forms (Brief Introduction)

Jordan Canonical Form (JCF)

Even when a matrix is not diagonalizable, it is always similar to a matrix in **Jordan canonical form**, which is a nearly diagonal matrix with Jordan blocks on the diagonal. It helps in analyzing non-diagonalizable systems.

Though not usually used directly in civil engineering, understanding JCF supports numerical analysis methods and control theory.

31.6 Applications in Civil Engineering

1. **Modal Analysis in Structural Engineering**
   - Involves finding natural frequencies and mode shapes.
   - Matrix similarity helps in reducing stiffness and mass matrices to diagonal form.
2. **Finite Element Method (FEM)**
   - Transformation of local stiffness matrices to global coordinates.
   - Coordinate transformations via similarity play a crucial role.
3. **Vibration Analysis**
   - Eigenvalue problems: natural frequency computation.
   - Similar matrices maintain the same spectral characteristics.
4. **Principal Stress and Strain Transformations**

- Stress/strain tensors are symmetric matrices.
- Rotation to principal axes uses orthogonal similarity (congruence).

5. **Solving Linear Differential Systems**
    - Reducing coefficient matrix to diagonal (or simpler) form for efficient solution.

31.7 Examples

Example 1: Checking Similarity

Let

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Are A and B similar?

Solution:

- A has eigenvalue $\lambda = 2$ of algebraic multiplicity 2.
- But the matrix A has only **one** linearly independent eigenvector:

$$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{Null space dimension} = 1$$

So, A is **not diagonalizable**, hence not similar to B (which is diagonal).

Example 2: Diagonalization Using Similarity

Let

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}$$

Find if A is diagonalizable.

Solution:

- Characteristic equation:

$$\det(A - \lambda I) = (4 - \lambda)(3 - \lambda)$$

- Eigenvalues: $\lambda_1 = 4, \lambda_2 = 3$

- Since eigenvalues are distinct, A is diagonalizable.
- Find eigenvectors:
  - For $\lambda = 4$: $(A - 4I)x = 0$
  - For $\lambda = 3$: $(A - 3I)x = 0$
- Construct matrix P with eigenvectors, compute $D = P^{-1}AP$

Hence, $A \sim D$, and similarity is established via diagonalization.

31.8 Orthogonal Similarity (Special Case)

If the change-of-basis matrix P is **orthogonal** ($P^{-1} = P^T$), the similarity is called **orthogonal similarity**:

$$B = P^T AP$$

Orthogonal similarity is especially useful for:

- Symmetric matrices (like stress/strain tensors)
- Preserving length and angle (important in mechanics)

31.9 Congruence vs Similarity (Advanced Insight)

In applied civil engineering contexts, particularly structural analysis, it's useful to distinguish **matrix similarity** from **matrix congruence**.

Matrix Congruence:

Two square matrices A and B are said to be **congruent** if:

$$B = P^T AP$$

Where P is an invertible matrix (not necessarily orthogonal).

> Note: This formula *looks similar* to orthogonal similarity, but congruence does **not preserve eigenvalues** — it preserves **quadratic forms**, which is more relevant in elasticity and structural mechanics.

Use in Civil Engineering:

- Stress-strain relationships: $\sigma = D\varepsilon$, where D is the stiffness matrix.
- Change of basis in such tensor equations uses **congruence**, not similarity.

31.10 Rational Canonical Form (for completeness)

Though not typically used in civil applications directly, **Rational Canonical Form (RCF)** gives a systematic way to classify square matrices up to similarity, especially over fields where eigenvalues may not exist (e.g., modular arithmetic).

RCF is useful in:

- Control systems theory.
- Theoretical linear algebra.

Given matrix A, there exists an invertible matrix P such that:

$$P^{-1}AP = R$$

Where R is a **block diagonal matrix** formed using **invariant factors** derived from the minimal polynomial of A.

It ensures:

- A unique canonical form for each similarity class.
- Useful in proving theoretical results, though not computationally efficient in practice.

31.11 Numerical Algorithms for Similarity Transformations

In computational civil engineering (e.g., finite element software), algorithms are used to compute similarity-based transformations.

Key Algorithms:

1. **QR Algorithm:**

    - Used to compute eigenvalues.
    - Based on repeated similarity transformations:

$$A_k = Q_k^T A_{k-1} Q_k$$

- where $A_k \to$ converges to an upper triangular matrix with eigenvalues on the diagonal.

2. **Schur Decomposition:**

    - Every square matrix is unitarily similar to an upper triangular matrix.
    - For real matrices, a **real Schur form** is used:

$$A = QTQ^T$$

- where Q is orthogonal and T is upper quasi-triangular.

3. **Jordan Reduction Algorithms:**
    - Rarely used in numerical practice due to instability.
    - Useful in symbolic computation environments (e.g., Mathematica, Maple).

These techniques are embedded in modern software like ANSYS, STAAD.Pro, and MATLAB used in civil engineering simulations.

31.12 Orthogonal Diagonalization of Symmetric Matrices

Theorem (Spectral Theorem):

If A is a **real symmetric matrix**, then:

- All eigenvalues of A are real.
- There exists an **orthogonal matrix** Q such that:

$$Q^T A Q = D$$

- where D is a diagonal matrix with eigenvalues of A on the diagonal.

This is fundamental in:

- Principal stress/strain calculations.
- Transformation to principal axes in mechanics.

Application: Principal Stresses

Given stress tensor:

$$\sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix}$$

The **principal stresses** are the eigenvalues of this matrix, and the **principal directions** (angles) are given by eigenvectors. Since σ is symmetric, it can be orthogonally diagonalized.

31.13 Similarity and Systems of Linear Differential Equations

For a linear system of ODEs:

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

If A is diagonalizable, say $A = PDP^{-1}$, then the solution becomes:

$$\vec{x}(t) = Pe^{Dt}P^{-1}\vec{x}(0)$$

Where:

- $e^{Dt} = \text{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$
- Solution is easy to compute when A is similar to a diagonal matrix.

This method is used in:

- Earthquake response of multi-storey buildings.
- Time-dependent dynamic analysis of structures.

31.14 Block Diagonalization via Similarity

For large systems, it's often useful to **reduce a matrix to block diagonal form** via similarity.

If A has invariant subspaces, then there exists an invertible matrix P such that:

$$P^{-1}AP = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}$$

Each block A_i acts independently on its own subspace — reduces computational load.

Use in FEM:

- In large stiffness matrices, symmetry and sparsity allow decomposition into smaller subdomains.
- Similarity transformations decouple the system for parallel processing.

31.15 Similarity over Complex Field

Sometimes, real matrices are not diagonalizable over $\mathbb{R}$ but are diagonalizable over $\mathbb{C}$. Example:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- Has complex eigenvalues $\pm i$
- Not diagonalizable over $\mathbb{R}$, but diagonalizable over $\mathbb{C}$

This has implications in:

- Harmonic motion
- Rotational dynamics (moment tensors)

Understanding similarity over $\mathbb{C}$ is necessary in advanced wave propagation, vibration, and circular motion analysis in civil engineering.
