# Chapter 5: Complex Exponential Function

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## Introduction

In civil engineering, the understanding of oscillatory and wave-like phenomena such as vibrations in structures, damping, and alternating currents is vital. These phenomena often involve solutions to differential equations with complex roots, which naturally lead to **complex exponential functions**. This chapter explores the concept, properties, and applications of the **complex exponential function**, which is fundamental in bridging real-world engineering problems with mathematical modeling, particularly in the context of differential equations and signal analysis.

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## 5.1 Euler's Formula

Euler's formula forms the cornerstone of the complex exponential function:

$$e^{ix} = \cos x + i \sin x$$

where:

- $i$ is the imaginary unit, $i^2 = -1$,
- $x$ is a real number (angle in radians).

**Interpretation:**

Euler's formula expresses a complex exponential as a combination of real trigonometric functions. It geometrically represents a point on the unit circle in the complex plane.

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## 5.2 Complex Exponential Function

The exponential of a complex number $z = x + iy$ is defined as:

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

**Key Points:**

- The real part $x$ controls the magnitude (growth/decay),
- The imaginary part $y$ determines the oscillation (angle/rotation),
- This function maps spirals in the complex plane.

## 5.3 Properties of the Complex Exponential Function

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$:

1. **Addition Rule:**

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$$

2. **Modulus:**

$$|e^z| = |e^{x+iy}| = |e^x(\cos y + i \sin y)| = e^x$$

3. **Periodicity:**

$$e^{z+2\pi i} = e^z \quad (\text{since } \cos(y + 2\pi) = \cos y, \sin(y + 2\pi) = \sin y)$$

4. **Derivative:**

$$\frac{d}{dz}e^z = e^z$$

5. **Multiplicative Inverse:**

$$e^{-z} = \frac{1}{e^z}$$

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## 5.4 Relationship with Trigonometric Functions

By Euler's identity:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

These identities allow us to **transform trigonometric expressions into exponential form**, making them easier to differentiate or integrate in complex analysis or signal processing.

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## 5.5 Polar Form of Complex Numbers and Exponential Notation

Any non-zero complex number $z = r(\cos\theta + i\sin\theta)$ can be written as:

$$z = re^{i\theta}$$

where:

- $r = |z|$ (modulus),
- $\theta = \arg(z)$ (argument or angle).

This form simplifies multiplication, division, and finding powers and roots of complex numbers.

**Multiplication:**

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

**Division:**

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

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## 5.6 De Moivre's Theorem

For any integer $n$:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Using Euler's formula:

$$(e^{i\theta})^n = e^{in\theta}$$

Useful in solving trigonometric equations, computing powers and roots of complex numbers in civil engineering problems involving harmonic motion.

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## 5.7 Application in Solving Linear Differential Equations

Second-order differential equations with complex roots lead to solutions involving the complex exponential function.

**Example:**

Solve:

$$\frac{d^2y}{dt^2} + 4y = 0$$

**Auxiliary equation:**

$$r^2 + 4 = 0 \Rightarrow r = \pm 2i$$

**General solution:**

$$y(t) = C_1 \cos(2t) + C_2 \sin(2t)$$

or

$$y(t) = Ae^{2it} + Be^{-2it}$$

The complex exponential solution can be converted to trigonometric form using Euler's formula, and is widely used in vibration and wave analysis in structural engineering.

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## 5.8 Graphical Representation

Plotting $e^{ix}$ on the complex plane gives a **unit circle**, rotating counterclockwise with increasing $x$. The exponential spiral $e^{x+iy}$ gives a **logarithmic spiral**, spiraling outwards as $x$ increases.

This has applications in:

- Modeling helical structures (like spiral staircases or springs),
- Analyzing rotating bodies,
- Solving PDEs in cylindrical coordinates.

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## 5.9 Real and Imaginary Parts of $e^{ix}$

The function $e^{ix}$ can be dissected into its **real and imaginary components**, which are the **cosine** and **sine** functions, respectively:

$$e^{ix} = \cos x + i \sin x$$

So:

- **Real Part:** $\text{Re}(e^{ix}) = \cos x$
- **Imaginary Part:** $\text{Im}(e^{ix}) = \sin x$

This breakdown is not just a theoretical formality—it has applications in **Fourier analysis**, where signals are decomposed into sine and cosine (or exponential) components.

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## 5.10 Periodicity and Rotations in the Complex Plane

**Periodicity:**

$$e^{i(x+2\pi)} = \cos(x + 2\pi) + i \sin(x + 2\pi) = \cos x + i \sin x = e^{ix}$$

This confirms the **periodic nature** of the complex exponential function in the imaginary part.

**Rotation:**

If $z = re^{i\theta}$, then multiplying by $e^{i\phi}$ rotates the point by angle $\phi$:

$$z \cdot e^{i\phi} = re^{i(\theta+\phi)}$$

**Engineering application:** This is analogous to rotating a force vector or stress tensor in mechanics or transforming between coordinate frames.

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## 5.11 Logarithm of a Complex Number

The **complex logarithm** is defined as the inverse of the complex exponential:

If $z = re^{i\theta}$, then:

$$\ln z = \ln r + i\theta$$

But here, $\theta$ is **multi-valued**, since:

$$e^{i\theta} = e^{i(\theta+2\pi n)}, \quad n \in \mathbb{Z}$$

So the logarithm is a **multi-valued function**:

$$\ln z = \ln r + i(\theta + 2\pi n)$$

**Principal Value:**

$$\text{Log } z = \ln r + i\Theta, \quad \text{where } \Theta \in (-\pi, \pi]$$

**Applications:**

- Used in solving equations involving exponential terms.
- Appears in **integration in the complex plane**.
- Important in **fluid flow problems** involving potential functions.

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## 5.12 Complex Powers and Roots

Given a complex number $z = re^{i\theta}$, we define its **complex power** as:

$$z^n = r^n e^{in\theta}$$

Similarly, the $n^{\text{th}}$ **roots** of a complex number are given by:

$$\sqrt[n]{z} = r^{1/n} e^{i(\theta + 2k\pi)/n}, \quad k = 0, 1, \ldots, n-1$$

**Geometric Interpretation:**

- The $n$ roots lie **evenly spaced** on a circle of radius $r^{1/n}$ in the complex plane.
- They form the **vertices of a regular polygon**.

**Application in Civil Engineering:**

- Used in the **modal analysis** of structures.
- Helps analyze **resonance frequencies** in mechanical systems.
- Relevant in **dynamic response** modeling of damped/undamped systems.

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## 5.13 Damped Harmonic Motion and Complex Exponentials

A general **damped oscillation** can be modeled using:

$$y(t) = e^{-\alpha t}(A\cos(\omega t) + B\sin(\omega t)) = \text{Re}\left(Ce^{(-\alpha + i\omega)t}\right)$$

Where:

- $\alpha$ controls **damping** (decay),
- $\omega$ is the **angular frequency**,
- $C$ is a complex constant.

This shows how complex exponentials **naturally arise** in second-order differential equations related to:

- Seismic vibrations in buildings,

- Suspension bridges,
- Structural response to dynamic loads.

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## 5.14 Fourier Series and the Role of $e^{inx}$

The **Fourier Series** expresses a periodic function $f(x)$ as a sum of sines and cosines or complex exponentials:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Where $c_n$ are **Fourier coefficients**.

This is widely used in:

- **Signal processing**,
- **Structural vibration analysis**,
- **Load modeling** over time in civil engineering simulations.

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## 5.15 Visualizing Complex Exponentials Using Argand Diagrams

In an **Argand diagram**, the complex exponential $e^{ix}$ traces the **unit circle**.

For a general complex exponential $z = e^{x+iy}$:

- $x$ controls the **radius** (scaling),
- $y$ controls the **angle** (rotation).

This is crucial in:

- Simulating **rotating systems**,
- Understanding **wave propagation**,
- Representing **phasors** in alternating current analysis.

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## 5.16 Engineering Use Cases Recap

| Use Case | Description |
|---|---|
| Vibration Analysis | $e^{i\omega t}$ form for modeling undamped/damped oscillations |
| Structural Dynamics | Solutions to beam vibrations and mass-spring-damper systems |

| Use Case | Description |
|---|---|
| AC Circuit Theory | Use of complex exponential for voltages and currents |
| Rotational Dynamics | Representation of rotation in 2D and 3D |
| Seismic Modeling | Time-dependent loads via Fourier components |
| Signal Transmission | Encoding and decoding signals in structures (e.g. bridge health monitoring) |

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## Exercises

1. Show that $e^{i\pi} + 1 = 0$ using Euler's formula.
2. Express $\cos(3x)$ in terms of exponential functions.
3. Solve $\frac{d^2y}{dx^2} + 9y = 0$ using complex exponentials.
4. Find all cube roots of $8(\cos\pi + i\sin\pi)$.
5. If $z = 5e^{i\frac{\pi}{4}}$, find $z^3$ and express the answer in Cartesian form.

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