Solid Mechanics
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Lecture - 22
Solving problems involving Extension-Torsion-Inflation

Hello everyone! Welcome to Lecture 22! In this lecture, we will solve some problems involving Extension-Torsion-Inflation.

1 Question 1: (start time: 00:27)

A cylinder with a variable cross section radius is considered as shown in Figure 1. The radius of the cylinder varies linearly from r_1 at the clamped end to r_2 at the other end. The length of the cylinder is L. A torque T is applied at the free end. How much is the rotation of the cross section at the free end?

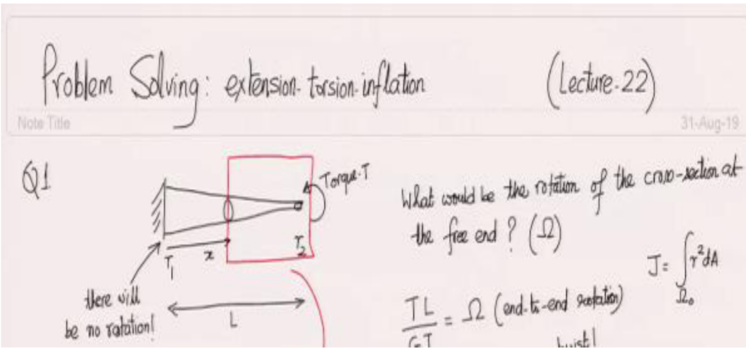

Figure 1: A torque T applied at the free end of a clamped cylinder with variable cross section radius.

Solution: (start time: 01:43) In the previous lecture, we had derived that the end-to-end rotation Ω is given by

$$\Omega = \frac{TL}{GJ} \tag{1}$$

but it holds only when the cross-section is uniform over the entire length. For the current case, a different form of the above equation is useful as shown below:

$$\Rightarrow T = GJ \underbrace{\left(\frac{\Omega}{L}\right)}_{\text{twist}(\kappa)} = GJ\kappa \tag{2}$$

where κ denotes twist or the rate of change of rotation of the cross section. As the same torque T acts in every cross-section (as shown below), the above equation implies that twist varies along the length as the polar moment of area J is varying here due to varying radius. For an even general case where the material of the tube as well as torque changes along the length, one can use the below formula to obtain twist at a given location:

$$T(x) = G(x)J(x)\kappa(x) \tag{3}$$

where the twist κ is related to cross-sectional rotation Ω through

$$\kappa(x) = \frac{d\Omega}{dx}. \tag{4}$$

The variation of T along the axis of the cylinder can be found by making a free body diagram. We cut a section of the cylinder at a distance x from the clamped end. The two cut parts are shown in Figure 2.

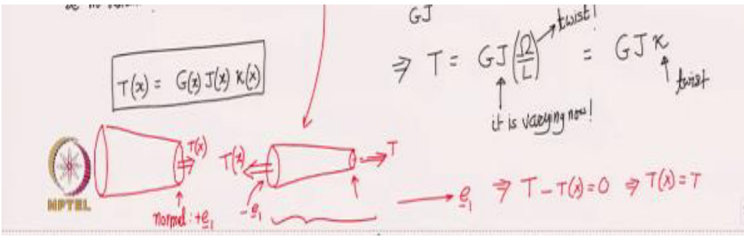

Figure 2: Two parts of a cylinder obtained after cutting a section at a distance x from the clamped end

The external torque T acts on the right end of the right section. There will be an internal torque on the other end of the right section also. As this cross-section has its normal along $-\underline{e}_1$, the torque $T(x)$ will act here in $-\underline{e}_1$ direction. Similarly, as the cross-section normal of the right end of the left section points in $+\underline{e}_1$ direction, the torque on it points along $+\underline{e}_1$ direction. We can notice that this convention also adheres to Newton's third law as the torque on the right end of left section and that on the left end of the right section form action-reaction pairs. We then do the moment balance for the right section. The total moment on this section should add up to zero, i.e.,

$$T - T(x) = 0 \Rightarrow T(x) = T. \tag{5}$$

This means that the torque at every cross-section is the same and equal to applied external torque T even though cross-sectional radius is changing here. Assuming that the cylinder is made up of a single material, its shear modulus will also be a constant, i.e.,

$$G(x) = G. \tag{6}$$

To find $J(x)$ for a cross-section of radius $r(x)$, we integrate over the area of the cross section, i.e.,

$$\begin{aligned} J(x) &= \int_{\Omega_0} r^2 dA = \int_0^{r(x)} \int_0^{2\pi} r^2 (r dr d\theta) = \int_0^{2\pi} d\theta \int_0^{r(x)} r^3 dr \\ &= 2\pi \frac{r^4}{4}\bigg|_0^{r(x)} = \frac{\pi r^4(x)}{2}. \end{aligned} \tag{7}$$

Upon plugging (5), (6) and (7) in equation (3), we obtain

$$\kappa(x) = \frac{T(x)}{G(x)J(x)} = \frac{2T}{G\pi r^4(x)}. \tag{8}$$

Using equation (4), the end-end rotation can be found by integrating twist along the axis of the cylinder, i.e.,

$$\begin{aligned} \Omega(L) - \Omega(0) &= \int_0^L \kappa(x)dx \\ \Rightarrow \Omega(L) &= \frac{2T}{\pi G}\int_0^L \frac{1}{r^4(x)}dx \quad \text{(left end of the tube is clamped)} \end{aligned} \tag{9}$$

As r varies linearly along the axis of the cylinder and

$$r(0) = r_1, \; r(L) = r_2, \tag{10}$$

the radius for a general cross-section can be found by linear interpolation of the above two points, i.e.,

$$r(x) = r_1 + \frac{r_2 - r_1}{L}x. \tag{11}$$

We can substitute this in equation (9) to get

$$\Omega(L) = \frac{2T}{\pi G}\int_0^L \frac{1}{\left(r_1 + \frac{r_2 - r_1}{L}x\right)^4}dx. \tag{12}$$

2 Question 2: (start time: 14:40)

A composite shaft is clamped at both its ends. It is composed of three sub-shafts as shown in Figure 3. The first part is made up of bronze (shear modulus G_B) and has length L_B and radius R_B. The second part is made up of aluminium (shear modulus G_A) and has length L_A and radius R_A. The third part is made up of steel (shear modulus G_S) and has length L_S and radius R_S. There is a torque T_1 acting at the interface of aluminium and bronze parts and a torque T_2 acts at the interface of aluminium and steel parts. Both T_1 and T_2 act along $+\underline{e}_1$ direction. What is the maximum shear component of traction in each of the three shafts?

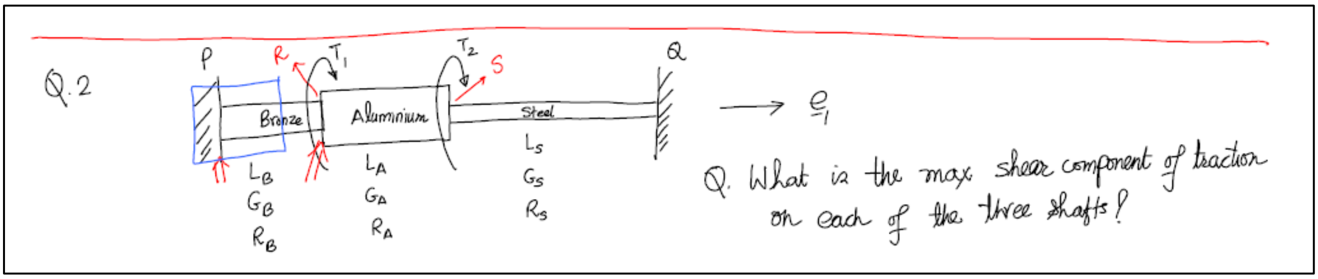

Figure 3: A composite shaft made up of bronze, aluminium and steel: torques T_1 and T_2 act on the shaft as shown.

Solution: (start time: 18:00) For finding shear component of traction, we need to find the shear strain for which we require the torque acting on each of the three shafts. In this problem, we will not be able to find the torques just by balance of forces and moments. This is a statically indeterminate problem as we will see when we try to solve it. Figure 4 shows the free body diagram of the entire shaft.

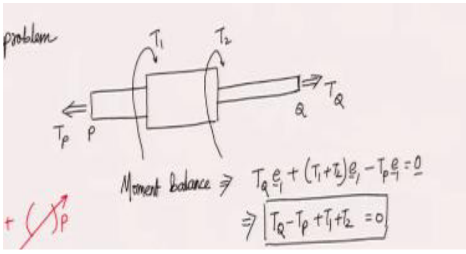

Figure 4: Free body diagram of the entire shaft.

The torques T_1 and T_2 are externally applied torques and point in the $\underline{e}_1$ direction while T_P and T_Q are reaction torques applied by the clampings on the shaft. The torque T_P points in the negative direction because the cross section normal at P points along $-\underline{e}_1$ direction. The moment balance of the entire shaft gives us

$$T_Q\,\underline{e}_1 + (T_1 + T_2)\,\underline{e}_1 - T_P\,\underline{e}_1 = 0$$

$$\Rightarrow \boxed{T_Q - T_P + T_1 + T_2 = 0} \tag{13}$$

As there are no forces acting on the shaft, force balance will not give us anything. So, we have one equation and two unknowns T_P and T_Q. Thus, this is a statically indeterminate problem. To solve such problems, we need to take deformation into account. We know that whenever equal and opposite torques are applied at different points in a beam, the beam undergoes torsion as shown in Figure 5. A straight line on the outer surface of the cylinder (parallel to the axis) becomes a helix after deformation due to twisting of the cylinder.

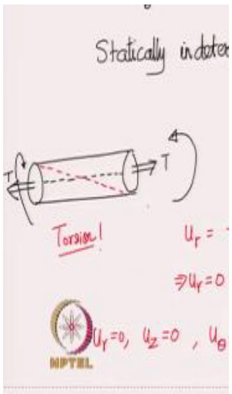

Figure 5: Equal and opposite torques on the end cross-sections of a cylinder causes torsion in the cylinder

We had derived the displacement function for the extension-torsion-inflation scenario and the result is summarized below:

$$\begin{aligned} u_r &= -\nu\epsilon_z r + \left(\frac{P}{2(\lambda+\mu)}\frac{r_1^2}{r_2^2-r_1^2}\right)r + \frac{P}{2\mu r}\frac{r_1^2 r_2^2}{r_2^2-r_1^2}, \\ u_\theta &= \frac{\Omega}{L}rz, \\ u_z &= \epsilon_z z. \end{aligned} \tag{14}$$

As there is no internal pressure applied, $P = 0$. The beam is clamped at the two ends and no forces are applied anywhere. So there is no axial strain ϵ_z either. Thus, for the pure torsion scenario, we get displacements as

$$u_r = 0, \quad u_\theta = \frac{\Omega}{L}rz, \quad u_z = 0 \tag{15}$$

We can note that torsion alone does not generate any radial or axial displacement. The strain matrix accordingly simplifies to

$$\underline{\underline{\epsilon}}_{(r,\theta,z)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{\Omega r}{2L} \\ 0 & \dfrac{\Omega r}{2L} & 0 \end{bmatrix} \tag{16}$$

The only non zero term is $\epsilon_{\theta z}$ which in terms of twist can be written as

$$
\begin{aligned}
\epsilon_{\theta z} &= \frac{1}{2}\underbrace{\frac{\Omega}{L}}_{\text{twist}} r = \frac{1}{2}\kappa r \\
\Rightarrow \gamma_{\theta z} &= \kappa r
\end{aligned} \tag{17}
$$

The stress matrix can then be written as

$$
\left[\underline{\sigma}\right]_{(r,\theta,z)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & G\kappa r \\ 0 & G\kappa r & 0 \end{bmatrix}. \tag{18}
$$

This is the stress matrix for pure torsion and it only has $\tau_{\theta z} \neq 0$. It is the state of stress at any point in the composite shaft. At a point, we have infinite planes. We need the plane on which the shear component of traction is maximum. In the lecture on Mohr's circle, we had discussed that for finding the maximum shear component, we need to obtain the difference of the maximum and minimum principal stresses and divide that by 2. To find the principal stress components, we can use the concept of Mohr's circle. As the first column of the stress matrix is all zero, $\underline{e}_r$ is a principal axis. Thus, we can draw the Mohr's circle for those planes whose normals are of the form:

$$
\underline{n} = \cos\theta\, \underline{e}_\theta + \sin\theta\, \underline{e}_z \tag{19}
$$

The center of the circle will be at the mid point of $\sigma_{\theta\theta}$ and σ_{zz} on the σ-axis. As both of them are zero, the center will be at the origin. We then mark the value of $\tau_{\theta z}$ (= $G\kappa r$) on the τ-axis. This point also corresponds to θ plane. With center and θ plane known, we can get the radius by joining these two points and equals $G\kappa r$. The Mohr's circle so drawn is shown in Figure 6.

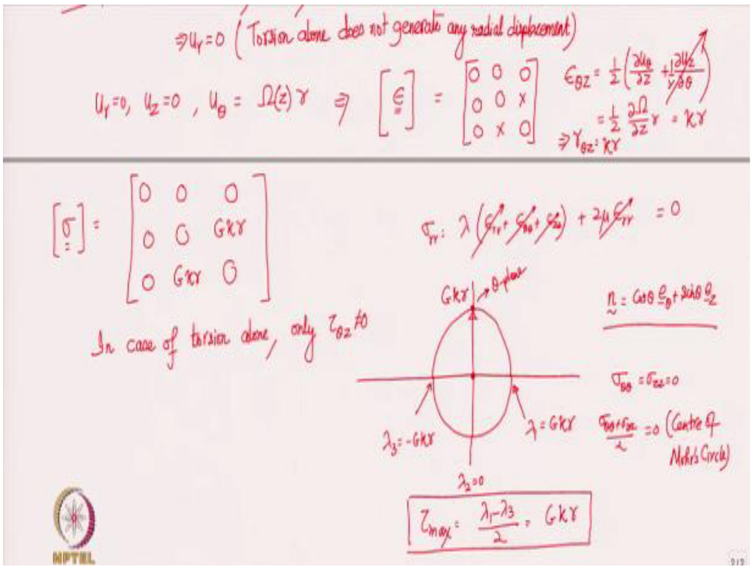

Figure 6: Mohr's circle for the state of stress given in (18).

We can now extract the principal stress components from the circle, i.e.,

$$\lambda_1 = G\kappa r, \qquad \lambda_3 = -G\kappa r \tag{20}$$

and λ_2 is zero corresponding to $\underline{e}_r$ principal direction. The maximum shear component of traction is thus:

$$\tau_{max} = \frac{\lambda_1 - \lambda_3}{2} = G\kappa r. \tag{21}$$

We have to find the maximum shear component of traction for each of the three sub-shafts. From our above analysis, it turns out to be $G\kappa r$ and r is maximized for points on the lateral surface. We also need to find the variation of twist along the axis for each of the three shafts. We had found in the previous problem (equation (3)) that torque at an arbitrary cross section is given by

$$T(z) = G(z)\kappa(z)J(z). \tag{22}$$

This time z denotes the coordinate along the axis. The twist κ is then

$$\kappa(z) = \frac{T(z)}{G(z)J(z)}. \tag{23}$$

As each sub-shaft is made of a single material and has a constant cross section radius, the denominator $G(z)J(z)$ is constant for the three sub-shafts separately. We just need to find the variation of torque in each of them. We first draw the free body diagram of a section of the bronze shaft. We make a cut in the bronze shaft and consider its left part as shown in Figure 7.

Figure 7: Free body diagram of the left part of the bronze shaft.

T_P acts as a reaction torque from the left clamping and $T(z)$ is exerted by the other part of bronze shaft. Balance of moment for this section gives

$$T_B = T_P \tag{24}$$

where T_B represents the torque within the bronze shaft and it does not vary within the bronze shaft. Now, we cut a section in the aluminium shaft such that bronze and aluminium shafts are included in the left part and draw its free body diagram (see Figure 8). This time we also have external torque T_1 acting in the free body diagram.

Figure 8: Free body diagram of the left part of composite shaft with a section cut in the aluminium part of the composite shaft

Moment balance gives torque in the Aluminium section (T_A) as

$$T_A = T_P - T_1. \tag{25}$$

Finally, we cut a section in the steel shaft such that all three materials are included in the left part. We draw the free body diagram of this left part as shown in Figure 9.

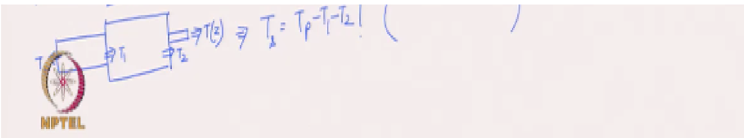

Figure 9: Free body diagram of the left part of the composite shaft when a section is cut in the steel part

Moment balance gives torque in the steel section (T_S) as

$$T_S = T_P - T_1 - T_2. \tag{26}$$

We can see that torque within each of the three shafts does not vary. But, we can also observe that

$$T_B \neq T_A \neq T_S \tag{27}$$

If we now come back to equation (23), all the quantities on the RHS are constant within each sub-shaft separately. Thus, twist (κ) is constant within each of the three sub-shafts. At the same time, its value is different for the three sub-shafts as their radii, G and T values are distinct. To find the maximum shear component of traction (= $G\kappa r$), we need the value of twist in the three regions. We have expressed the torques in the three sub-shafts in terms of T_P which is still an unknown. To find this unknown T_P, we need to use the constraint that the two ends of the shaft are clamped. So, both the ends do not undergo any rotation and the end-to-end rotation for the composite shaft must be zero. Thus, the sum of the end-to-end rotations for the three sub-shafts should be zero. This is the extra equation that required taking deformation into account. Using equation (1), the end-to-end rotations for each of the sub-shafts will be

$$\Omega_R - \Omega_P = \frac{T_B L_B}{G_B J_B}, \quad \Omega_S - \Omega_R = \frac{T_A L_A}{G_A J_A}, \quad \Omega_Q - \Omega_S = \frac{T_S L_S}{G_S J_S}. \tag{28}$$

Adding the three equations , we finally get

$$\begin{aligned} \Omega_R - \Omega_P + \Omega_S - \Omega_R + \Omega_Q - \Omega_S &= \frac{T_B L_B}{G_B J_B} + \frac{T_A L_A}{G_A J_A} + \frac{T_S L_S}{G_S J_S} \\ \Rightarrow \Omega_Q - \Omega_P &= \frac{T_B L_B}{G_B J_B} + \frac{T_A L_A}{G_A J_A} + \frac{T_S L_S}{G_S J_S} \end{aligned} \tag{29}$$

However, $\Omega_Q - \Omega_P$ is zero since it equals end-to-end rotation of the doubly clamped composite shaft. We can now substitute T_B, T_A and T_S from equations (24), (25) and (26) into (29) to obtain

$$\begin{aligned} & \frac{T_P L_B}{G_B J_B} + \frac{(T_P - T_1) L_A}{G_A J_A} + \frac{(T_P - T_1 - T_2) L_S}{G_S J_S} = 0 \\ \Rightarrow \quad & T_P \left(\frac{L_B}{G_B J_B} + \frac{L_A}{G_A J_A} + \frac{L_S}{G_S J_S} \right) = \frac{T_1 L_A}{G_A J_A} + \frac{(T_1 + T_2) L_S}{G_S J_S} \end{aligned} \tag{30}$$

We can solve this equation for T_P. Once we get T_P, we can find torque in the three sub-shafts from equations (24), (25) and (26) and hence twist in them using equation (23). Then, we can use equation (21) to calculate the max shear component of traction in the three sub-shafts. We need to use the outer radius of the three sub-shafts to get the maximum shear component.