Discrete Mathematics
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Lecture – 51
Hamiltonian Circuit

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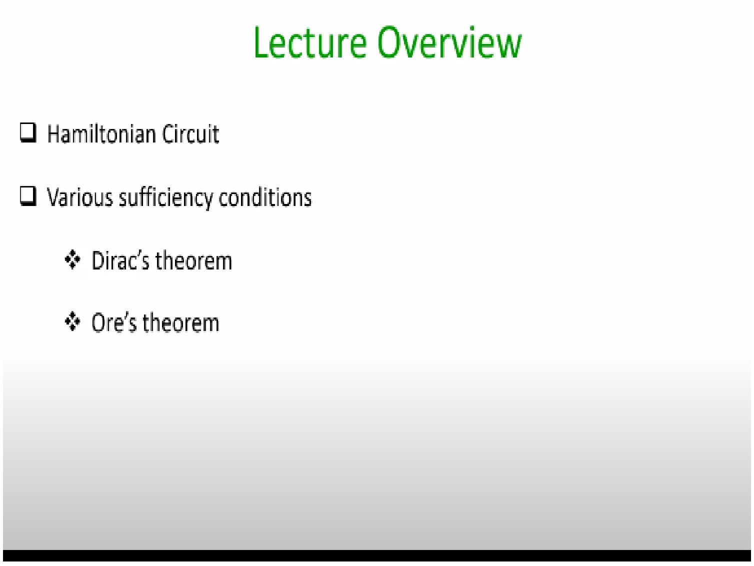

Hello everyone welcome to this lecture and the plan for this lecture is as follows. So, in this lecture we will discuss about Hamiltonian circuit. And we will discuss about some sufficiency conditions for the existence of Hamiltonian circuit in a graph namely the Dirac's theorem and Ore's theorem.

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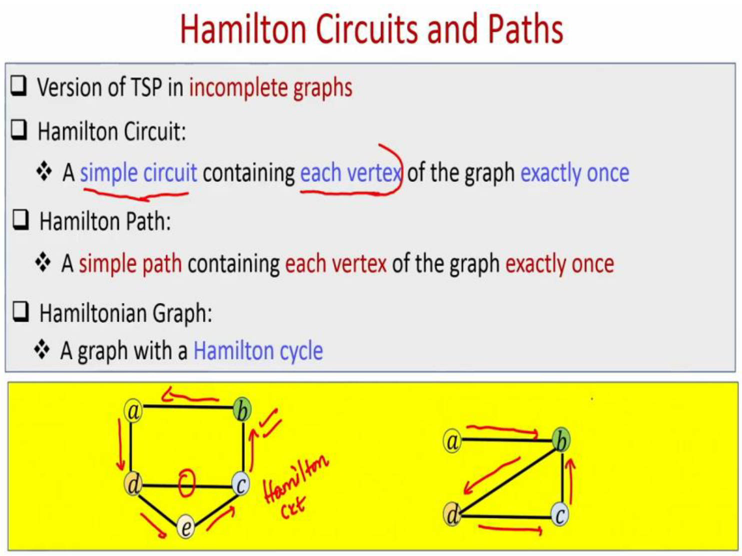

So, what is a Hamilton Circuits and Hamilton Paths? So, on a very high level it is a version of travelling salesman problem in incomplete graphs. So, specifically a Hamiltonian circuit is a simple circuit. So, what do we mean by a simple circuit? It is a tour which starts and ends at the same vertex and all the edges are distinct. But it is a special type of simple circuit in the sense that the vertices are not allowed to be repeated.

And every vertex of the graph occurs exactly once that means no vertex of your graph is missed it will appear definitely that does not mean that all the edges of the graph are covered. So, this is different from your Euler circuit. The Euler circuit, the requirement first at all the edges should be covered as part of your tour. Here the requirement is that all the vertices should be covered. And no vertex should be repeated.

Whereas Hamiltonian path is a simple path that means it may not start and end at the same vertex. And it should cover exactly once each vertex of the graph. We call a graph as a Hamiltonian graph if it has at least one Hamiltonian cycle. So, for instance if I consider the first graph here this, this graph has a Hamiltonian circuit because if I make a tour like a to d, d to e, e to c, c to b and b to a then it covers all the vertices.

So, now you can see that this edge not there as part of the tour that is fine; the requirement is that you should start and end at the same vertex and traverse each vertex of the graph exactly once.

Whereas this graph, so this is the first graph it has an Hamiltonian circuit whereas the second graph it does not have a Hamiltonian circuit, why? Suppose I start my tour at a go to b and go to d and go to c and then go to b but I cannot repeat the edge between a to b because that is already traversed.

So that means it will be violating my requirement of a simple circuit. So, edge will be repeated, so I will start my tour at a and my tour at b but I have covered all the vertices but my starting point and end points are different and hence this is an Hamiltonian path and not an Hamiltonian circuit.

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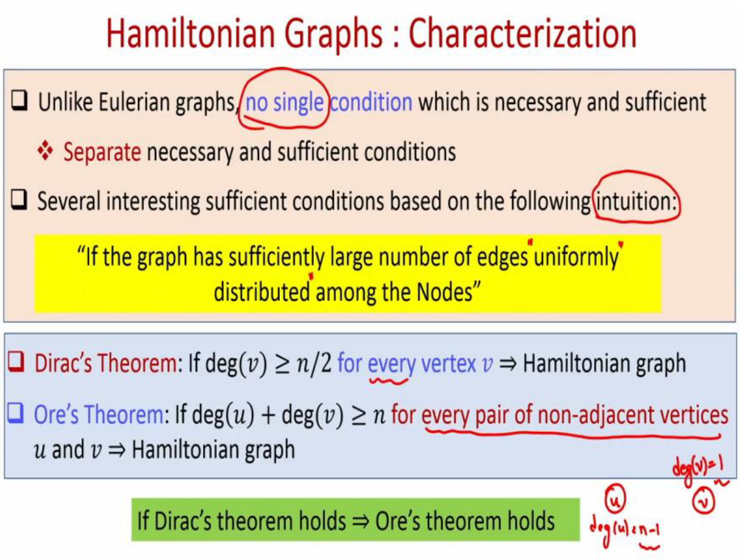

So, now the next interesting question will be is there a necessary and sufficient condition through which we can check whether a given graph is Hamiltonian or not. Unfortunately like Eulerian graphs where we have a single condition which was both necessary as well as sufficient. So, just to recall we had the condition that all the vertices of your graph should have even degree and that was both necessary as well as sufficient for the existence of an Eulerian circuits.

But that is not the case for Hamiltonian graph we do not have a single condition which is simultaneously necessary as well as sufficient. And we do have separate conditions which are either necessary but not sufficient or sufficient but not necessary. So, what we will discuss; we will discuss in this lecture 2 important sufficient conditions for the existence of Hamiltonian graph. I stress that those 2 conditions are not necessary conditions.

So, both those interesting sufficient conditions are based on the following intuition. The intuition is that if your graph is such that it has sufficiently large number of edges and those edges are uniformly distributed among the nodes, so it is a very vague term uniformly distributed. So, imagine it is uniformly distributed among the nodes and the graph is very dense. Then we can argue about the existence of the Hamiltonian circuit.

So, let us see what do we mean by uniformly distributed in 2 different contexts. So, the first sufficiency condition is what we call us Dirac's theorem. So, it says that if you have a connected graph where the degree of every vertex is at least $\frac{n}{2}$, I stress for every vertex then it is guaranteed that your graph is Hamiltonian. However you can quickly verify that this condition is not necessary that means you may have a graph which is Hamiltonian where even though the degree of all the vertices is not $\frac{n}{2}$.

A very simple graph could be your cycle graph which is Hamiltonian where the degree of every vertex is 2 which need not be $\frac{n}{2}$. So, the Dirac's theorem says that if your graph is sufficiently dense in the sense that degree of every vertex is $\frac{n}{2}$ then the graph is Hamiltonian. Whereas another related sufficiency condition is Ore's condition which says that if your graph is such that for every pair of non adjacent vertices u and v the summation of the degree of u and v is at least n then your graph is a Hamiltonian graph.

So, again what Ore's condition says is that you take every pair of non adjacent edges and some of their degrees it should be at least n. So, if you compare the Dirac's condition and Ore's condition then it is easy to see that if Dirac's condition holds in your graph that means if the degree of every vertex in the graph is at least $\frac{n}{2}$ then you take any pair of non adjacent vertices u, v and some of their degrees it will be at least n.

So that means if Dirac's condition is ensured in your graph then that also ensures that Ore's condition is also ensured in the graph but the other way around may not be true. You may have a pair of vertices u and v, say which are non adjacent and say where the degree of u is n - 1 and say where the degree of v is 1. So, in total the sum of their degrees is greater than equal to n. But you will see that u is taking the bulk of the degree, whereas v is taking a very small degree namely one but Ore's condition says that even in this case the graph is Hamiltonian. So, in that sense Ore's condition is more flexible it does not put too much restriction in terms of degrees on the vertices of the graph. It says that you take any pair of non adjacent vertices as long as you guarantee that the summation of their degrees is n it does not matter how exactly n is distributed as degrees across the degree of u and v.

It could be the case that both of them are $\frac{n}{2}$, $\frac{n}{2}$ or u is taking say $\frac{2}{3}$ of n and v is taking $\frac{1}{3}$ of n and so on, still my graph will be Hamiltonian. Whereas Dirac's condition is slightly stringent in the sense that demands that every vertex should have degree $\frac{n}{2}$ and then only I can argue that my graph is Hamiltonian. I stress here that none of these 2 conditions is a necessary condition because as I said if you take the cycle graph. Then the graph is Hamiltonian, you take cycle graph of any number of vertices it will be Hamiltonian graph but neither the Dirac's condition holds not the Ore's condition holds.

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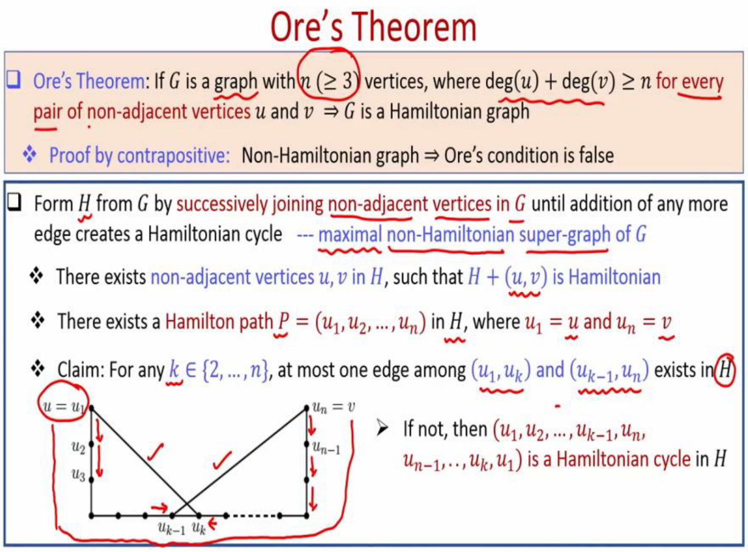

So, what we will prove here we will just discuss a very high level overview of the proof of Ore's theorem. So, what we want to argue here is the following you are given a graph which has at least 3 vertices because then only it makes that to talk about the summation of degrees of non adjacent pair of vertices. So, it is given that you take any pair of vertices u and v which are non adjacent the sum of their degrees is at least n.

Then I want to argue that my graph is indeed Hamiltonian. And the proof will be by contrapositive that means we would not give you a concrete algorithm, by running which you can find out your Hamiltonian circuit or Hamiltonian tour but rather we will argue logically that indeed this implication is true. That means we are trying to give a non constructive proof where we are arguing proof by contrapositive. So, what will be the proof by contrapositive? So, we will argue that if your graph G is not Hamiltonian then Ore's condition is false. Now what does it mean? When I see all Ore's condition is false. So, Ore's condition is a universally quantified statement. It says that for all u, v which are non adjacent that summation of degree of u and v is greater than or equal to n.

So, when I say that negation of this Ore's condition because that is what will be the 'q' part in the proof by contrapositive. So, the negation of the Ore's condition is that there exists, at least one pair of u and v for which the Ore's condition is not satisfied. So that is what we will argue here; if the graph is non Hamiltonian then there exists at least one pair of non adjacent vertices u and v for which the Ore's condition is false.

I am not going to argue that for every u, v pair in my graph the Ore's condition is false because that is not what we mean by the negation of the Ore's condition. And proof is very clever here. So, we will first transform the graph G to another graph H. And intuitively what is this graph H: it is the maximal non Hamiltonian super graph of G. Namely this graph H will have my original graph G and it might have some additional edges as well we will see how those additional edges are added.

It may be the case that your graph H is the same as the graph G itself but it may not be the case. So, if that is not the case then that means I have expanded my graph G and got a new graph H and the new graph H will be still non Hamiltonian and it will be maximal we will see soon what exactly I mean by maximal here. So, the way I construct my graph H from the graph G is the following I keep on successively joining non adjacent pair of vertices in my graph G that means I take my graph G.

So, imagine this is your graph G, I randomly choose a non adjacent pair of vertices in the graph G. So, u and v and I check whether by adding the edge between the node u and v I get a Hamiltonian cycle in the graph or not. If by adding the edge between u and v in my graph G I still do not obtain any Hamiltonian cycle in the graph then I will add that edge, that dummy edge, and then I keep on repeating this process.

And keep on adding more and more edges by identifying non adjacent pair of vertices and checking whether by adding an edge between those 2 non adjacent vertices whether I get an Hamiltonian cycle or not. If I do not get then again add that edge and keep on bombarding more and more edges keep on adding more and more edges in the graph G till you reach a saturation point, saturation point in the sense that you reach a point, where you identify a pair of vertices u and v which are non adjacent such that if in the super graph H which you have obtained till now by keep on adding more and more edges you add this new edge between the vertex u and v you get an Hamiltonian cycle. If you reach that point that means that is the saturation state and you should stop that means you should not now add the edge u and v.

That means you have now identified a critical pair of vertices u and v, critical in the sense that now if you add the edge between these non adjacent vertices u and v in the graph H you will get a Hamiltonian cycle you stop at that point. And now the proof of this proof by contrapositive will be focusing on this u and this v that means this non adjacent u non adjacent v and we will argue that with respect to this specific u and v the Ore’s condition is false.

That means if I sum up the degrees of u and v it would not be greater than equal to n. So that is the proof idea. So, the proof basically tries to identify a critical pair of non adjacent vertices u and v such that by adding the edge between u and v in the super graph you are bound to get a Hamiltonian cycle. So, in that sense my graph H maximal that means I cannot add more edges in the graph edge beyond this point because I have the guarantee of existence of at least one non adjacent pair of critical vertices.

So, since I have now identified my node u and v which are the critical vertices what I can now say is the following. I can say that in my super graph H, I do have a Hamiltonian path. That Hamiltonian path will start with the node u and it will end with the vertex v and it will cover all the vertices of the graph. Why it is a Hamiltonian Path? Because of the fact that if I would have added the edge between the nodes u and v in my graph H then that edge if I traverse or include in my path P that would have given me a Hamiltonian cycle.

So, because of that fact I can conclude that there exists a Hamiltonian path in my graph H starting with u and ending with v which is simple and which covers all the vertices of the graph. So that is why u_1 to u_n. Now here comes a very crucial claim. We claim here that if you forget about the starting point of the tour namely u or u_1 and then focus on the remaining vertices namely the vertex 2, vertex 3, vertex 4, vertex n which occurs in your Hamiltonian Path P then for each such vertex u_k you cannot have simultaneously the edges (u_1, u_k) and edge (u_{k-1}, u_n) existing in your super graph H.

So, remember this claim is with respect to the super graph, namely the graph which you have obtained by keep on adding or by expanding your original graph G, this claim is not about your original graph G; this claim is about the super graph H. So, pictorially what I am arguing here is so this is your Hamiltonian Path P. So, let me draw it in a better way. So you started with u_1 then u went to u_2, u_3 and like that u_{k-1}, u_k, u_{n-1}, u_n where u_n is v and u_1 is u and all the vertices are bound to appear here.

So, my claim is if you take any k so for instance if I take k = 2 then the claim is that you cannot have simultaneously the edges (u_1, u_2). So, definitely (u_1, u_2) is there in your super graph H because that is why it is a part of your Hamiltonian Path P. So, the claim says that since you have (u_1, u_2) in the graph you cannot have the edge (u_n, u_1) in the graph. Because if I substitute k = 2 here it says that you cannot have simultaneously (u_1, u_2) as well as (u_1, u_n).

So, since you have the edge between u_1 and u_2 you cannot have the edge between u_n to u_1. Similarly the claim says that either you can have the edge between u_1 and u_3 in the graph H or you can have the edge between u_2 and u_n that is the claim here. So, let us prove this claim the proof is very simple and elegant. So, again the proof is by contradiction so imagine there is some intermediate k such that you simultaneously have the edge between u_{k-1} and u_n that means this edge is there in the graph.

As well as you have the edge between u_1 and u_k in the graph H again I stress that the claim is with respect to the super graph H. So, imagine both these edges are there then I can argue that actually from this Hamiltonian Path P I can extract out a Hamiltonian circuit in my super graph H itself which is a contradiction because as per my construction the super graph H is still non Hamiltonian.

So, how can I extract out a Hamiltonian circuit, it will be just like doing a crossover. So that means my extracted Hamiltonian circuit will be that you go from u_1 to u_2, you go from u_2 to u_3 you go all the way to u_{k-1} as per the Hamiltonian path but in the Hamiltonian path after u_{k-1} you have traversed to u_k. But what I could have done actually is the following: if the edge between u_{k-1} and u_n is there I could have followed this edge.

And then follow the rest of the tour as per the Hamiltonian path. That means go from u_n to u_{n-1} and u_{n-1} to u_{n-2} come back all the way to u_k. And since you have also the edge between u_k to u_1 because that is what I will obtain by assuming a contrary of this claim statement. I could follow this edge and end my tour at u itself and I would have covered all the vertices.

So that means the proof of the claim is based on the fact that if you have both these edges then by just crossing those 2 edges and following the rest of the things as part of your Hamiltonian path you can extract out a Hamiltonian circuit in your super graph H which is a contradiction. So, we have proved this claim what exactly this claim says?

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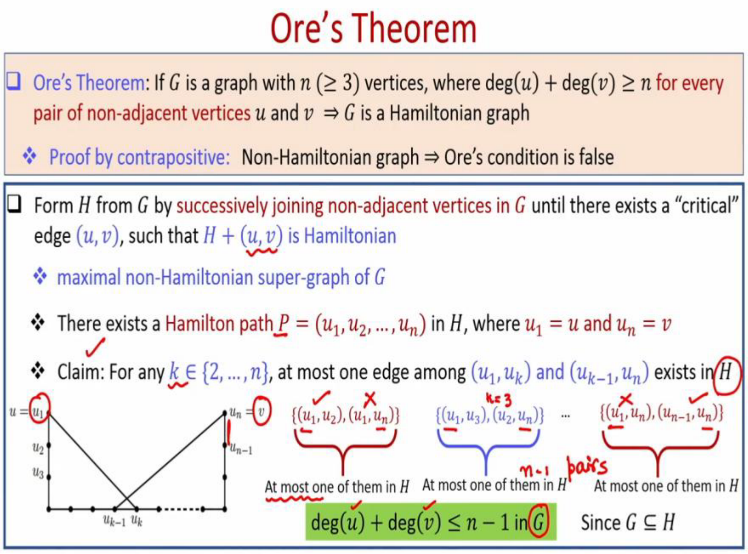

The claim says the following if I bunch or if I iterate over k = 2 to n and then for pair of vertices so for k = 2 the claim says that between u_1 and u_2 you can have the edge or between u_1 and u_n you can have the edge you cannot have both the edges u_1, u_2 as well as u_1, u_n that is not possible. At most one of these 2 edges is there in the graph H. The claim statement for k = 3 says that either you can have the H between u_1 and u_3 or you can have the edge between u_2 and u_n but you cannot have both these 2 edges simultaneously.

And similarly the claim for k = n implies that you can have either the H between u_1 and u_n or an H between u_{n-1} to u_n you cannot have both the edges simultaneously. Now that gives you the implication that remember my $u = u_1$ and $u_n = v$. That gives me the implication that in the super graph H the summation of the degrees of the vertices u and v is at most n - 1 why at most n - 1? Because I have n - 1 pairs here.

And in each pair I get only one edge guaranteed to be there in my super graph; in none of the pairs I cannot have both the pairs of edges present in my super graph H. So, for instance I know definitely the edge between u_1 to u_2 is there because that is why the Hamiltonian path P there that means the edge between u_1 and u_n is not there. And similarly I know that the edge between u_n and u_{n-1} is there.

Because that is the part of the Hamiltonian path; that means the edge between u_1 and u_n is not there. So, the way I have collected up the pair of vertices here I am counting the degrees of u_1. And the degrees of u_n, the degree of u_1, the degree of u_n, the degree of u_1, the degree of u_n and through each pair I either get one to the degree of u_1 or one to the degree of u_n through none of the pairs I can get simultaneously a degree for u_1 as well as a degree for u_n.

And how many such pairs are there: I have n - 1 such pairs. And as a result if even if I take the best case that means through each pair, remember the claim says that is it is not the case that from each pair definitely an edge is there in the super graph H; it says at most one edge. So, even if I take the best case for the graph H and through each pair I extract out or if my graph H is such that through each pair I get a guarantee that exactly one edge is there in my super graph H.

I can at most get a summation of the degrees of u and v to be exactly n - 1. And this is the case in my super graph H remember all the claims everything I made with respect to my super graph H. Now if the claim is true for my super graph H the claim is obviously true in my sub graph G because the nodes u and the nodes v were they are in my graph G I have not changed my vertex set.

When I expanded my graph G I kept intact my vertex set it is only the edge set which I kept on modifying that means I kept on adding edges between non adjacent pair of vertices till I identify a critical pair of non adjacent vertices. And in the super graph H I have proved that the degree of u and v if I take the summation of their degrees is at most n - 1. So, it is obviously the case that even the summation of the degrees of u and v in my original graph will be at most n – 1, because some of the edges between some of the edges incident on the node u and incident on the node v in my super graph edge might be because of the expansion process those edges may not be necessarily there in my original graph. So, since I have proved the statement for my super graph that shows the statement is obviously true even for my base graph G. So that is the simple not simple but that is our overall idea for the proof of Ore's theorem.

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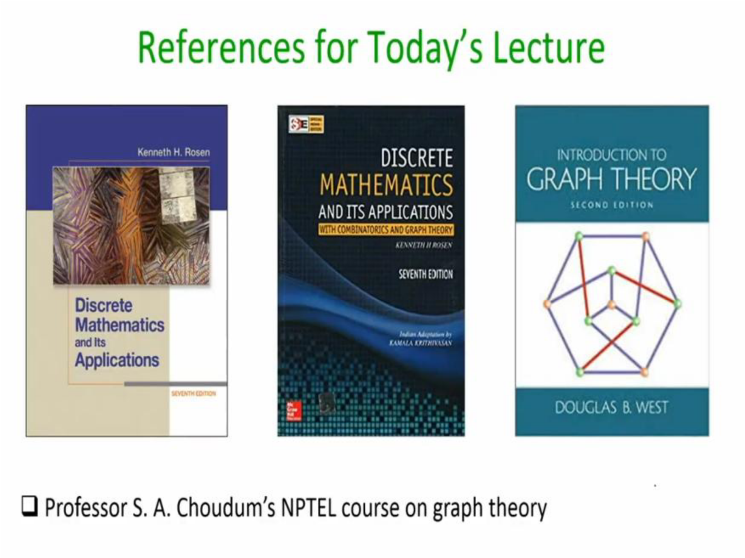

So that brings me to the end of this lecture. So, these are the references used and just to summarize: In this lecture we introduced a definition of Hamiltonian circuits and Hamiltonian path and unlike Euler graphs, Euler circuits, Euler path where we are we have a single condition which is both necessary and sufficient for the existence of Euler circuit and Euler path we do not have a single condition which is both necessary as well as sufficient for the Hamiltonian circuit and Hamiltonian path. So, we have seen 2 interesting sufficiency condition namely Dirac's condition and Ore's condition. Thank you.