Hydraulic Engineering
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Lecture # 59
Introduction to wave mechanics

Welcome students to the last module of hydraulic engineering course, in this module we are going to study in inviscid flow, the typical application that is wave mechanics, we are going to study linear wave theory, the derivation of velocity potential from scratch, we will also look at the boundary value problems and so, do you know to get started with that, we will first see what the linear wave theory is let us start with that.

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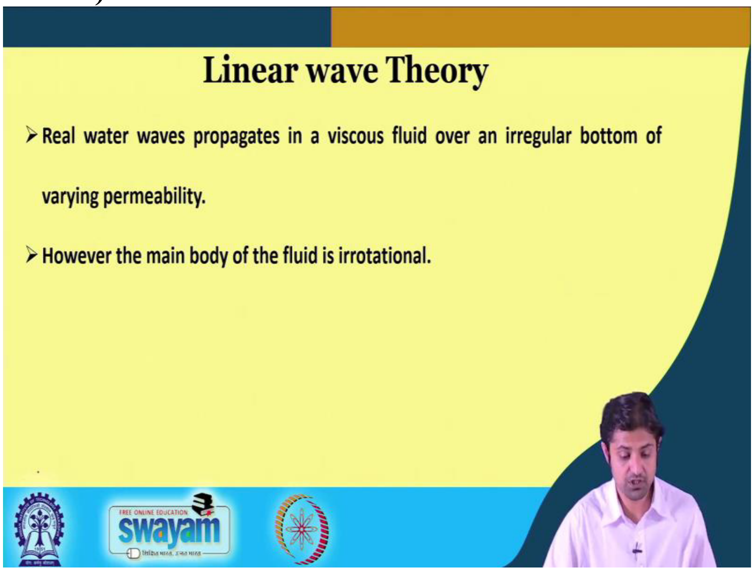

So, linear wave theory, what is that as the name indicates, the linear wave theory can be guessed as you know when the waves are considered linear in nature. So, the background is as everybody knows that the real water waves have you seen waves in the ocean. So, the disturbances that traveling in the ocean are actually waves they might not be linear but they are waves. So, the real water waves they propagate in viscous fluid. So, the way the fluid in which it propagates is viscous like water mostly propagates in water.

So, and you know water is a viscous fluid and it propagates over an irregular bottom of varying permeability so, that is a real case scenario. However, we assume that the main body of the fluid is a rotational because the viscous effects but we have learned in the; our laminar and turbulent and viscous fluid flow classes is that the viscosity with the viscosity effects are limited near to the bottom. So, the bottom above that bottom part, the main body of the fluid is in rotational.

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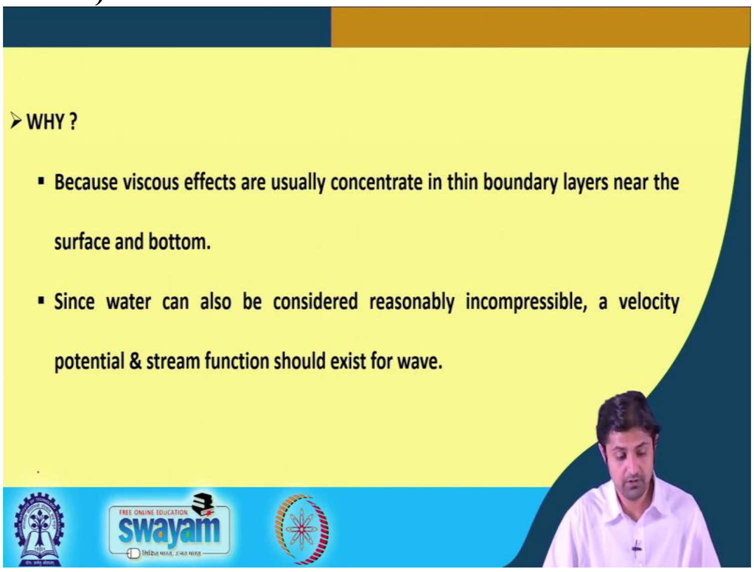

And what is the reason because, as I said viscous effects usually concentrate in thin boundary layers near the surface and also the bottom because the boundary layer will also be there at the surface. Since water can also be considered reasonably incompressible we can assume the existence of a velocity potential and stream function. so, going back to the basics you remember if we can say that water is incompressible then there will exist a velocity potential and stream function for an invested in irrotational flow case a recall your lectures from viscous fluid flow.

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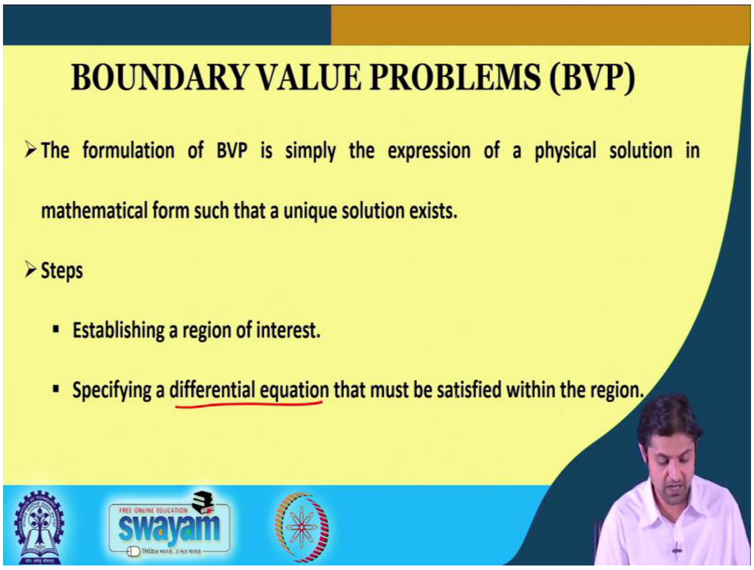

So, to before we go ahead and start learning about linear wave theory there is a very important concept that is called boundary value problems. This is not only important for this particular chapter, but any problem that has something to do with boundary values for example, we saw in the last week lectures and CFT that boundary values are some of the things that must be specified for the computation to start. So, we look in this part more closely at these boundary value problems.

So, the formulation of boundary value problem is simply the expression of a physical solution in a mathematical forms such that a unique solution exists. So, what happens is if there is an equation that is valid for entire domain entire area. And then there could be infinite number of solutions to that problem? For example, let us say there is y squared + x squared = + z squared = let us say 9.

So very simple to assume one of the solutions could be x = 0, y = 0 and z = 3 this is one solution. The other solution could be x = 3, y = 0 and z = 0 the another solution could be x = 0, x = 0, y = 3 z = 0 and similarly there could be many other combinations as I written down in form of the you know, integers, x, y z are integers, but there could be many solutions. But, boundary value problems the formulation of a boundary value problem is important. So, that there will exist only a unique solution per 1 particular value of boundary.

this equation will have 1 unique solution there cannot be any solution let us say, if we specify for x square y square + that z squared = 9 and sort of a boundary value we say just you know x, if we are given that y is you know 3, then it will have only even unique solution in terms of real number x = 0 and z = 0. I mean this we cannot exactly so, this is a boundary condition but trying to give an analogy that because of this boundary value there will exist a unique solution for a mathematical formulation otherwise there could be in finite number of solutions.

Now, what are the steps to formulate the boundary value problems? First is we have to establish a region of interest? So, where are we going to apply that boundary very prominent we have to determine for example, a region of interest would be a wave tank or tank where the water flows? Secondly, we have to specify a differential equation that must be satisfied within the region.

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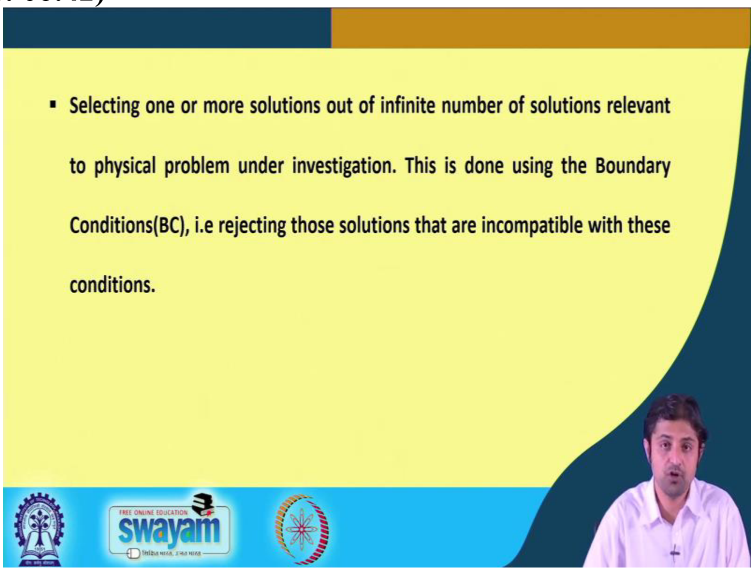

Thirdly, we have to select 1 or more solutions out of infinite number of solutions which are relevant to the physical problem under investigation. So, if there is an equation here if we specified a differential equation in the domain then the solution that have different of that differential equation could be infinite there could be many solutions,, but we have to select only one or more solutions to the physical problem under investigation and this is done using the boundary condition.

This is exactly what I told you a couple of minutes ago. So, we have to reject those solutions that are incompatible with these conditions. So, this actually is called the implementation of boundary condition. So, these steps repeat to repeat again is the first step is we have to specify a region of interest. Secondly, we write down the mathematical equation or a differential equation that must be satisfied in the entire region domain.

And third one is that we have to select only 1 or you know more solutions let us say 1 2 or 3 solutions out of infinite number of solutions that are relevant to the physical problem the real problem and this is done by using boundary condition very simple example is let us say a river for example, river or ocean or wherever and what we say a part of river or ocean, the channel. So, this is something like this, where the flow is coming like this, we know for sure, as we already read in our CFD class.

That continuity equation and navier stokes equation written by NS equation is; are to such differential equation that must be valid in this domain. Now, this will have in finite number of solutions. So, we must provide a boundary condition a boundary condition like that the water entering here as a velocity let us say 3 meters per second here there is a wall, you know, this is open boundary, we can either specify a top 3 surface or even a wall here.

So, what I mean to say these are the boundary condition that will determine that there will exist a unique solution. Now, in addition to the spatial boundary condition as I said so, this is let us say inlet, outlet wall for example. So, these are the; you know some here velocity is given like 10 meters per second or something. So, apart from this these are called the spatial boundary conditions or geometrical boundary condition, but there will also be a temporal boundary condition.

So, boundary condition in time that boundary condition in time is called initial condition. So, initial condition is nothing but a boundary condition in time. So, for example, we say at time t = 0, the entire fluid was at rest. So, this means that v = 0 at t = 0 becomes our initial condition and this should also be specified.

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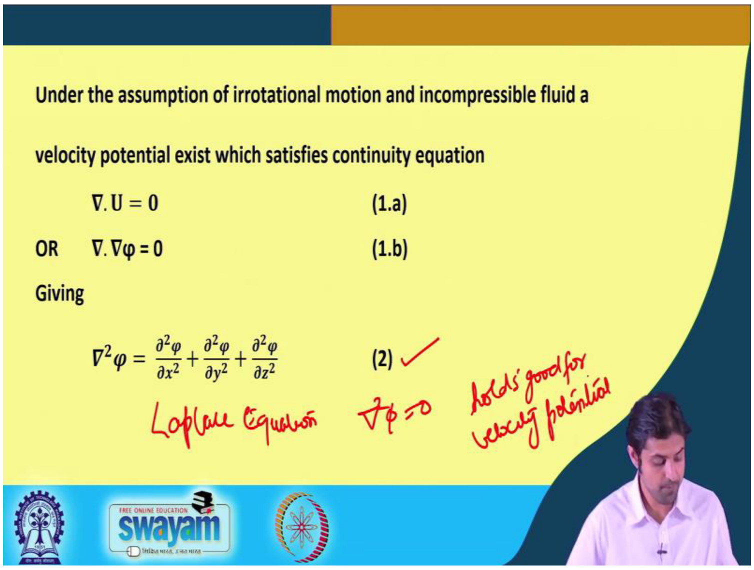

Now, under the assumption of irrotational motion and in compressible fluid as learned in our viscous fluid flow class and the basics of fluid mechanics kinematics that there will exist a velocity potential which should satisfy the continuity equation. So, what are the assumption irrotational motion and incompressible fluid there exists the velocity potential phi from our basic fluid mechanics classes that means

$$\nabla . \mathbf{U} = \mathbf{0}$$

let us call this equation 1.a or we know that if there exists a velocity potential ϕ, then U can be written as delϕ.

So putting this here we will get $\nabla . \nabla \phi = 0$ and this is nothing but

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

the call this equation number 2. So, if you follow we have got $\nabla^2 \varphi = 0$.

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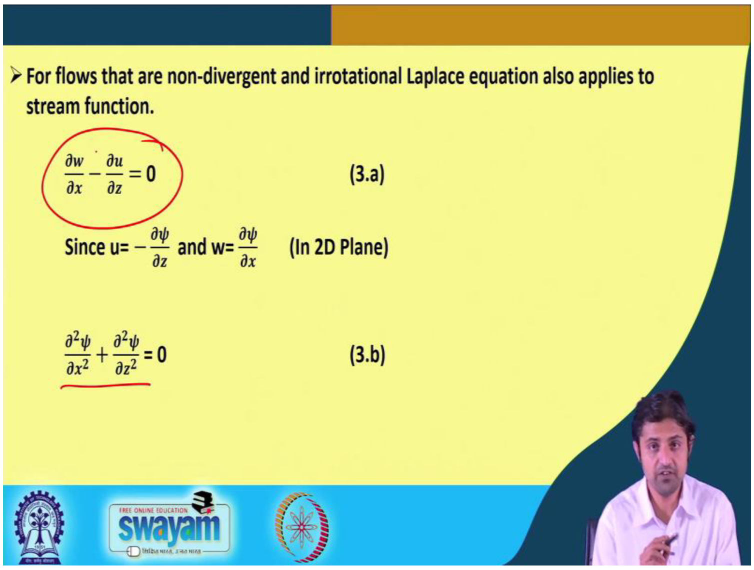

So, if we consider the flows which are non-divergent and irrotational. So, the previous equation was nothing but Laplace equation so, for a non-divergent and a rotational Laplace equation also applies to the stream function. So, we have seen for the velocity potential Laplace equation which is del squared phi = 0 is Laplace equation holds good. But, so, for the flows that are non-divergent and irrotational, we say that Laplace equation can also be applied to stream function.

How we are going to see, we know that for irrotational flow we have Del w by Del x if we consider of flow in u, so, x and z direction there are 2 directions that we assume there is no flow in y direction. So, the stream function as it you know can exist only in 2d can be written as Del w by Del x – Del u by Del z because the flows are rotational. Now, using the definition of a string function if u is - del phi del z and w is del phi del x and if we put this so we say del w by del x - del u by del z = 0 instead of w if we put here.

So it becomes Del del phi Del x of - u is here, so it becomes Del u Del z by Del z = 0 so it becomes so this is with - phi. So, it becomes del squared phi by del x squared + del Squared phi del z squared = 0 and this is what in 2d plane this is all again Laplace equation. So repeating that so, this gives us Laplace equation into 2d which given by equation number 3 b.

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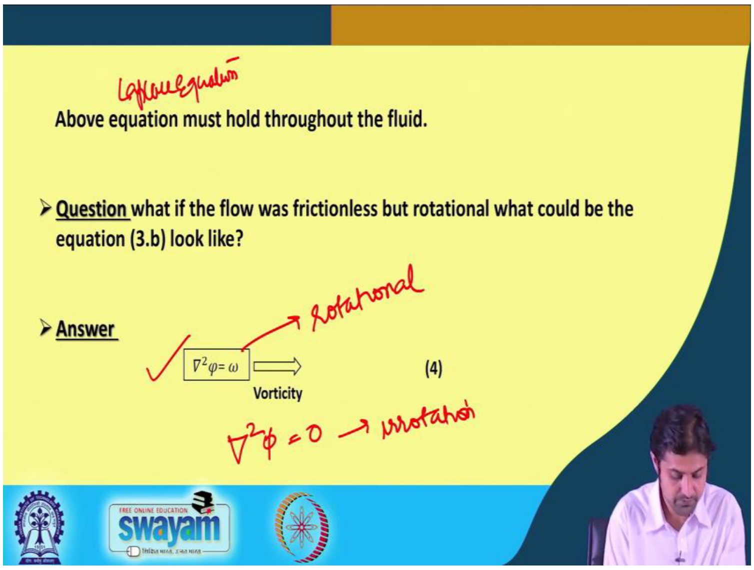

Now, the above equation as I said if you remember the boundary condition analysis, we said that the differential equation that we have formed should be valid in the entire domain of the fluid. So, in the current analysis, what are the 2 equations what are the equations that we have devised that holds true in the fluid using continuity equation, we have said that this equation Laplace equation above equation that is Laplace equation must hold throughout the fluid. So, at this point I have a small question to you think about it what if the flow was frictionless.

But rotational not a irrotational what could be the what could the equation 3 be look like. So, what could be the; this equation if you brush up your mind for irrotational flow this term was 0. But if it is rotational then it is given by vorticity w so, Del squared phi is written as omega which = the vorticity. So, this will be the resulting form of equation now for irrotational flow regards 0 and this is true for rotational flow.

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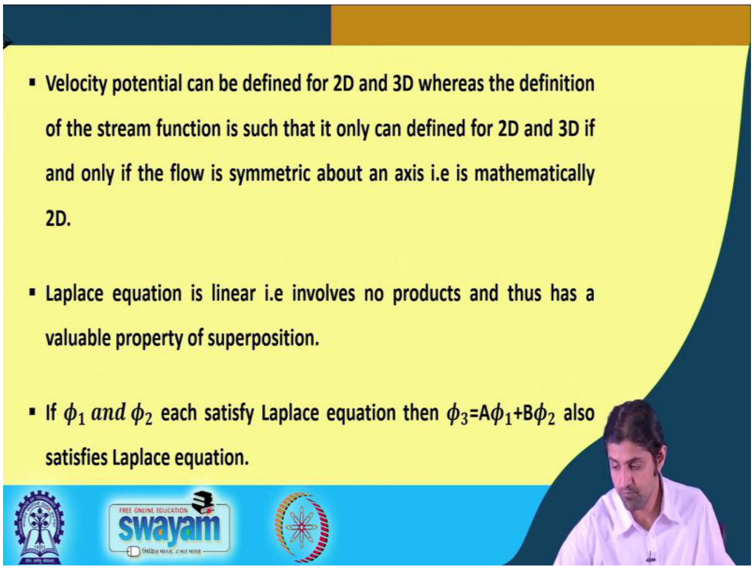

Now we know that the velocity potential can be defined for 2d and 3d, 2d means 2 dimension and 3d 3 dimension where as the definition of the stream function is such that it only can be defined for 2d and 3d if and only if the flow is symmetric about x axis. So we know velocity potential is valid for both 2 dimensions and 3 dimensions. However, the string function is valid only in 2 dimension. Our special case in which we can still write it in 3d is when the flow is symmetric.

Let us say there is a flow in x and y plane, this is x, this is y and this is z., so it is happening, but say this is so large that the variation in y direction is not there. So, actually the flow is symmetric. Therefore, mathematically it will again be called as 2d because the flow is symmetric about an axis and therefore, we can still stream function for such type of flows. The second thing to note is that Laplace equation is linear, that is it involves no products and thus has a valuable property of superposition.

So, we do not have terms like delta x delta y or phi 1 in you know, does not is not there. So, we try we in a different sense we say that Laplace equation is linear because it does not have any product. Therefor we can apply value the; you know the property of superposition to Laplace equations. So, what is that the superposition if we say phi 1 and phi 2 are the velocity potentials and each of them satisfy Laplace equation, then we can define another velocity potential phi 3.

Which are a combination of phi 1 and phi 2 that is A phi 1 + B phi 2 can will also satisfy the Laplace equation. So, any combination of phi 1 and phi 2 will satisfy Laplace equation a phi 1 and phi 2 each satisfy Laplace equation. The reason is that Laplace equation is linear and it does not involve any products and therefore can be super imposed.

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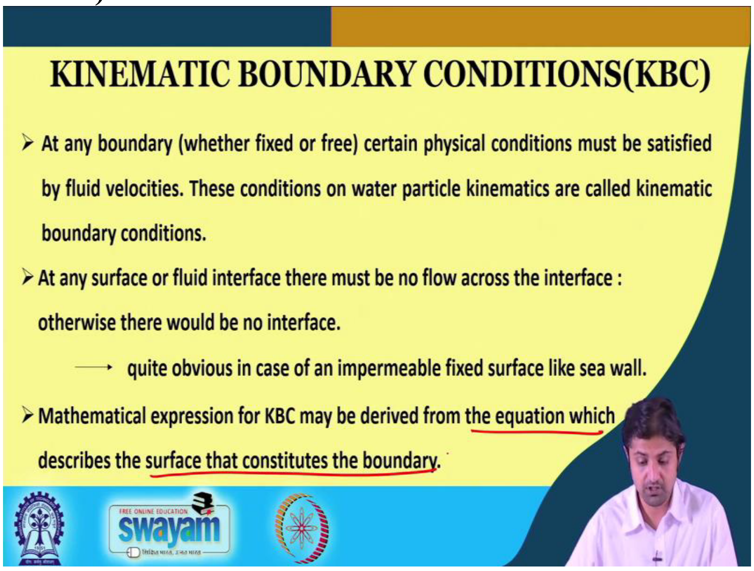

So, one important property is kinematic we are talking about boundary conditions. So, one of the important properties called kinematic boundary conditions, what is our KBC at any boundary whether fixed or free, certain physical conditions must be satisfied for fluid velocities. And these conditions on water particle kinematics are called dynamited boundary condition. At any surface or fluid interface, these for example, this is one of the example at any surface or fluid interface there must be no flow across the interface for example, if there is no ground here.

The boundary kinetic conditions will be the velocity W is going to be 0 in the downward direction or if there is a water surface here also the water will not have any velocity otherwise, if it has a velocity then it will still be water there then there is not going to be an interface. So, at any surface or fluid interface, there must be no flow across the interface and that is very, standard universal type of dynamic boundary condition.

So, this is quite obvious actually in case of impermeable fixed surface like sea wall mathematical expression of dynamic boundary condition may be derived from the equation, which describes the surface that constitutes the boundary. So, now, we talked what the current dynamic boundary condition is now, we have to find what the mathematical equation for such boundary conditions are and as said, these kinematic boundary conditions can be derived from were derived from the equation of the surface that constitutes the boundary. So, if we define a surface, we can determine the dynamic boundary condition we will understand this with an example.

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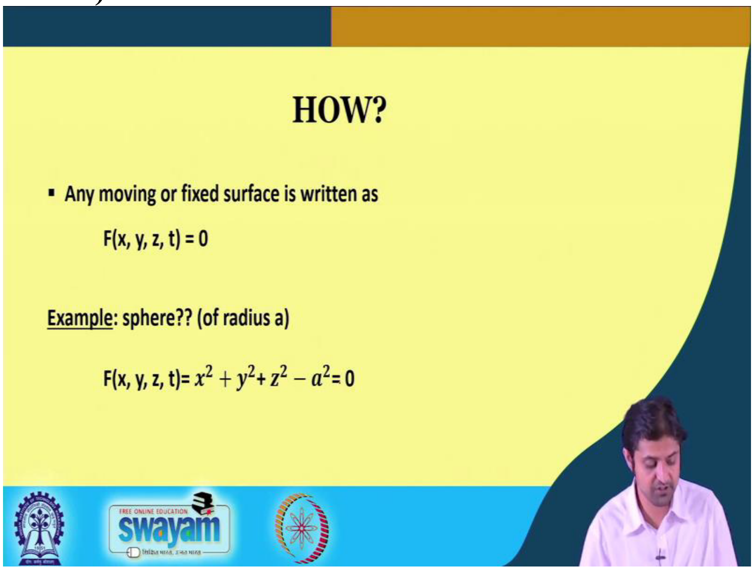

How? So we say that any moving or fixed surface, let us see if it is retained by so this is the equation of the surface F how it is dependent on x y z and t, we do not know, but let us say there is a function f x y z, t = 0. So let us take a real life example of sphere. A sphere let us say as a radius a So F of x y z, t can be written as x squared + y squared + x squared - a squared = 0. Let us say this is the surface of the sphere.

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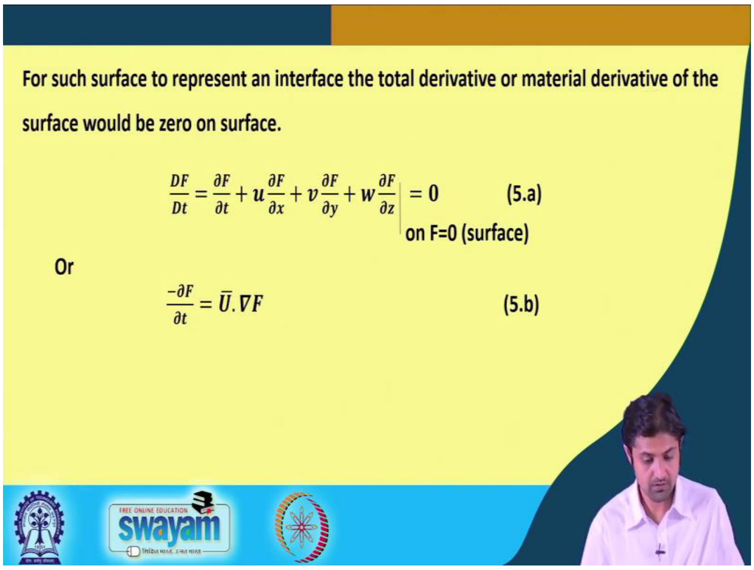

So for such surface to represent an interface, the total derivative or material derivative of the surface would be 0 on that surface. This is what our definition is that there should be no flow across the interface, which means that the total derivative or material derivative of the surface would be 0 on the surface because there is no flow across that. So, what we do is capital DF dt material derivative can be written as Del F del t + u del F del x + v del F del y + w del F del z let me use a laser pointer.

So,

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$

and what does this say the total derivative at the surface so on F = 0 which means surface should be 0 here or in other words we can say because this is 0. So, if we bring it on this side, we can write

$$\frac{-\partial F}{\partial t} = \bar{U}.\nabla F$$

a representation in a vector form. So, this is an important equation.

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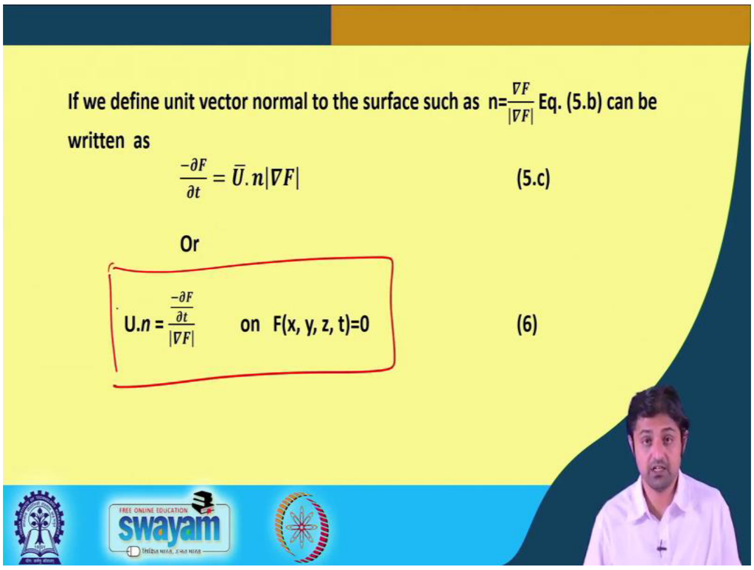

Now if we define a unit vector normal to the surface so unit vector normal to the surface can be written as,. So, normally if you recall your mathematics class this is a unit vector which is normal to the surface using this and we can write equation 5 b this thing as - delta F del F del t = u dot simply what we did instead of delta F I will just so, if you use this equation we just substituted n mod delta F we substituted this in equation 5 b and we get this value.

Or we can write, if we want to write u dot n will be nothing but if we just we what we do is we just bring this being the denominator. So, we get an equation u dot n = - Del F del t by mod of delta F on F x y z t = 0 u is a vector and n is a normal vector.

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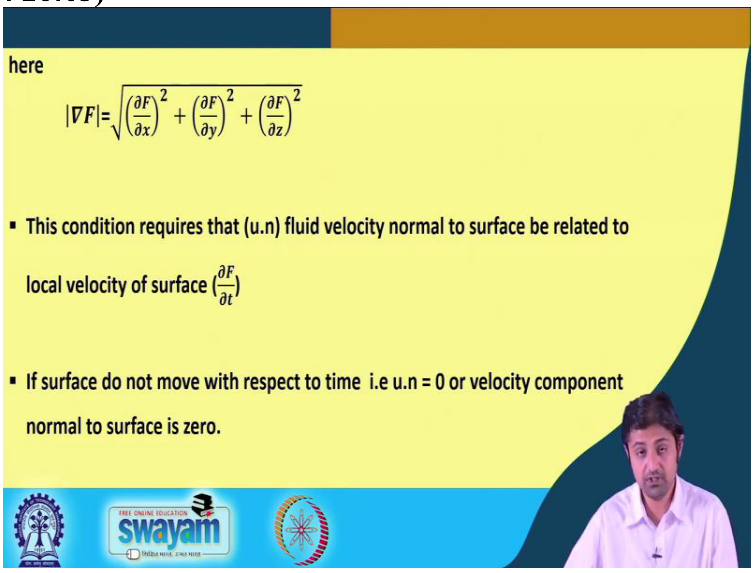

So, here delta F is nothing but simple delta F by delta x whole squared + delta F by delta y wholes squared + delta F delta z whole squared. Now, if you look at the condition here u dot n = - del F del t by mod of F, this condition requires that u dot n that is the fluid velocity normal to the surface this is u dot n correct should be related to the local velocity of the surface del F del t is the local velocity of the surface by this equation.

the one that we wrote before let us say if the surfers do not move, that means that a fixed, with respect to time that is the local velocity of the surface is 0 that means, u dot n was - del F del t by mod of F if it does not move then this is 0 which means u dot n = 0 or in other words the velocity component which is normal to the surface is 0. And we have seen this that when we assume a fixed surface like the riverbed or the seabed this you remember I drew a diagram and said the velocity here will be 0. We just used to say that but now we have proved this. That the velocity component normal to the surface will be 0.

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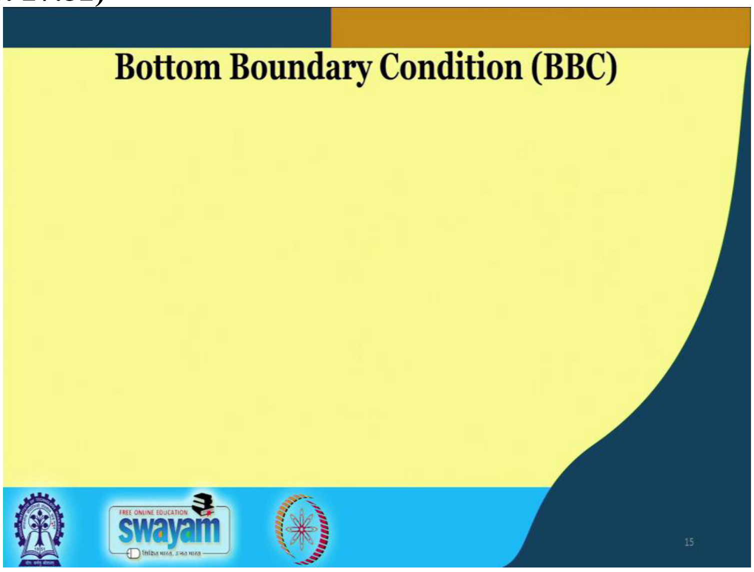

Now, we start with the bottom boundary condition. I think this is a Fine place to start. In our next lecture. We will start with this topic called the bottom boundary condition. Until that time. Thank you so much. I will see you in the next lecture.