Chapter 12: Dirac Delta Function

Introduction

The Dirac delta function, though not a function in the classical sense, is a powerful mathematical tool widely used in engineering, physics, and particularly in civil engineering applications involving differential equations, structural analysis, and signal processing. It is employed to model idealized point loads, impulses, or concentrated effects, enabling the formulation and solution of problems involving such phenomena. This chapter delves into the definition, properties, and applications of the Dirac delta function, providing a rigorous understanding tailored for civil engineering contexts.

12.1 Concept and Informal Definition

The **Dirac delta function**, denoted by $\delta(x)$, is not a conventional function but a **generalized function** or **distribution**. It is defined informally by the following properties:

- **Zero everywhere except at the origin:**

$$\delta(x) = 0 \quad \text{for } x \neq 0$$

- **Infinite at the origin:**

$$\delta(0) = \infty$$

- **Integral over the real line equals one:**

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

These properties make $\delta(x)$ useful for modeling idealized point effects.

12.2 Heuristic Interpretation

The Dirac delta function can be thought of as the limit of a sequence of functions that become increasingly narrow and tall, while keeping the area under the curve constant at 1. Some common approximations include:

12.2.1 Rectangular Approximation

$$\delta_\epsilon(x) = \begin{cases} \frac{1}{\epsilon}, & |x| < \frac{\epsilon}{2} \\ 0, & \text{otherwise} \end{cases} \quad \text{as } \epsilon \to 0$$

12.2.2 Gaussian Approximation

$$\delta_\epsilon(x) = \frac{1}{\sqrt{\pi\epsilon}} e^{-x^2/\epsilon} \quad \text{as } \epsilon \to 0$$

These are not the delta function itself but converge to it in the sense of distributions.

12.3 Sifting Property

One of the most important properties of the Dirac delta function is its **sifting** or **sampling** property:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a)$$

This means $\delta(x-a)$ "picks out" the value of $f(x)$ at $x = a$. This is useful in civil engineering when evaluating the effect of a point load or impulse at a specific location.

12.4 Properties of Dirac Delta Function

12.4.1 Even Function

$$\delta(-x) = \delta(x)$$

12.4.2 Scaling Property

$$\delta(ax) = \frac{1}{|a|}\delta(x), \quad a \in \mathbb{R}, a \neq 0$$

12.4.3 Shifting Property

$$\delta(x-a) : \text{concentrated at } x = a$$

12.4.4 Multiplication by a Function

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

12.4.5 Integration Involving Delta Function

$$\int_a^b f(x)\delta(x-c)\,dx = \begin{cases} f(c), & \text{if } a < c < b \\ 0, & \text{otherwise} \end{cases}$$

12.5 Derivative of the Dirac Delta Function

The **derivative** of the delta function $\delta'(x)$ is also a distribution, defined through its action on a test function $f(x)$:

$$\int_{-\infty}^{\infty} f(x)\delta'(x-a)\,dx = -f'(a)$$

This is useful in modeling sudden changes in systems (e.g., velocity jumps from an impulse in dynamics).

12.6 Delta Function in Higher Dimensions

In civil engineering problems involving spatial domains, the delta function extends naturally to multiple dimensions:

$$\delta(\vec{r} - \vec{r_0}) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$$

This allows modeling of point loads or sources in 2D or 3D structures.

12.7 Use of Dirac Delta in Civil Engineering Applications

12.7.1 Modeling Point Loads in Beam Theory

In structural analysis, the delta function is used to represent a concentrated load P applied at a point $x = a$:

$$q(x) = P\delta(x-a)$$

Where $q(x)$ is the distributed load on a beam. This simplifies solving the differential equation of deflection:

$$EI\frac{d^4y}{dx^4} = q(x) = P\delta(x-a)$$

12.7.2 Green's Functions

The delta function serves as the source term in **Green's function** techniques for solving boundary value problems in engineering.

12.7.3 Impulse Load in Dynamics

When a structure is subjected to an impulsive force at time $t = t_0$, it is modeled as:

$$F(t) = F_0 \delta(t - t_0)$$

This appears in vibration analysis, where equations of motion involve impulsive forces.

12.8 Dirac Delta as a Distribution (Advanced View)

To handle the delta function rigorously, it is treated as a **distribution** that acts on test functions:

Let $\phi(x)$ be a smooth test function. Then,

$$\langle \delta(x - a), \phi(x) \rangle = \phi(a)$$

This formalizes the sifting property and allows differentiation and integration of generalized functions.

12.9 Delta Function in Fourier Transforms

In signal processing and structural vibration analysis, the delta function appears naturally in the context of Fourier transforms:

12.9.1 Fourier Transform of Delta Function

$$\mathcal{F}\{\delta(x)\} = 1, \quad \text{and} \quad \mathcal{F}^{-1}\{1\} = \delta(x)$$

12.9.2 Delta Function from Inverse Transform

For example:

$$\int_{-\infty}^{\infty} e^{i\omega x} \, d\omega = 2\pi \delta(x)$$

This identity is used extensively in system response calculations.

12.10 Practical Examples

Example 1: Point Load on a Simply Supported Beam

Let a point load P be applied at the center of a simply supported beam of length L. The load function is:

$$q(x) = P\delta\left(x - \frac{L}{2}\right)$$

This leads to the governing differential equation:

$$EI\frac{d^4y}{dx^4} = P\delta\left(x - \frac{L}{2}\right)$$

Solving this using appropriate boundary conditions gives deflection $y(x)$.

Example 2: Impulse in Structural Dynamics

An impulse F_0 applied at time $t = 0$ to a mass-spring system gives:

$$m\frac{d^2x}{dt^2} + kx = F_0\delta(t)$$

Solution involves convolution with the system's impulse response function.

12.11 Relationship with Unit Step Function (Heaviside Function)

The Dirac delta function is closely related to the **Heaviside step function**, which is defined as:

$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

The delta function is the **derivative of the Heaviside function** in the distributional sense:

$$\frac{d}{dx}u(x) = \delta(x)$$

This relationship is particularly useful in modeling **sudden application of loads** in structural systems, such as the instant engagement of a support or switch in system behavior.

12.12 Representation of Discontinuous Functions

A function with a discontinuity at a point $x = a$ can be expressed in terms of the delta function:

$$f'(x) = \text{regular derivative} + [f(a^+) - f(a^-)]\delta(x - a)$$

This concept is used to describe sudden changes in system behavior, such as:

- A sudden jump in shear force in a beam,
- A pressure drop in a fluid system,
- A crack or joint in materials.

12.13 Laplace Transform of Dirac Delta Function

In civil engineering, Laplace transforms are used for solving time-dependent differential equations. The Laplace transform of the delta function is:

$$\mathcal{L}\{\delta(t - a)\} = e^{-as}, \quad a \geq 0$$

This property allows the modeling of instantaneous forcing terms in dynamic analysis of civil systems, such as:

- An impulse from a falling object on a structure,
- Switching on a mechanical load,
- Sudden impact or blast effects.

12.14 Convolution with the Delta Function

Convolution is a key operation in system response theory. The delta function acts as an identity in convolution:

$$(f * \delta)(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t - \tau)\, d\tau = f(t)$$

This implies that the delta function preserves the shape of signals or functions in linear time-invariant (LTI) systems.

In civil engineering:

- Convolution is used in **structural vibration analysis**,
- To determine **response functions** to arbitrary loading,
- To model **real-time feedback** in control systems.

12.15 Application in Soil Mechanics and Geotechnical Engineering

In geotechnical engineering, concentrated forces or loads are frequently applied to soil surfaces or foundations. The delta function can be used to model:

- **Point loads on elastic half-space**, leading to Boussinesq's solutions,
- **Vertical stress distribution** beneath loaded areas.

Example:

For a vertical point load P applied at the origin on the surface of a semi-infinite elastic medium (soil), the load can be modeled as:

$$\sigma_z(x, y, 0) = P\delta(x)\delta(y)$$

This allows deriving stresses and displacements using integral transforms and potential theory.

12.16 Application in Fluid Mechanics and Hydrology

The delta function is used in modeling:

- **Instantaneous pollutant injection** into a stream or groundwater:

$$C(x, t) = M\delta(x - x_0)\delta(t - t_0)$$

- **Impulse response of fluid systems**, like surge tanks or pipes,
- **Sudden rainfall events** in hydrologic modeling.

It simplifies equations involving conservation laws with impulsive source terms.

12.17 Application in Transportation and Traffic Flow

In modeling traffic systems:

- **Delta functions represent sudden vehicle entry/exit** on roads.

- For example, a single vehicle entering a road at time $t = t_0$ can be written as:

$$\rho(x, t) = \delta(x - x_0)\delta(t - t_0)$$

- **Traffic flow equations** then incorporate such impulses to simulate real-world events.

12.18 Use in Finite Element Method (FEM)

The delta function concept is foundational in the **weak form** formulation of differential equations for the finite element method.

In FEM, the test function approach uses:

$$\int_{\Omega} \delta(x - x_0)\phi(x)\, dx = \phi(x_0)$$

This allows precise application of **nodal loads** and ensures correct weighting in the element stiffness matrix. Delta functions also appear when defining **point sources** in meshless methods.

12.19 Computational Representation

Although the delta function is theoretical, numerical methods approximate it using:

- Very narrow Gaussian or rectangular functions,
- Kronecker delta in discrete systems:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

In computational civil engineering:

- FEM and FDM use **discrete delta approximations** for localized loading.
- Simulation software like ANSYS or ABAQUS internally applies this logic for **point forces**, **boundary conditions**, and **singularities**.

12.20 Limitations and Physical Interpretation

While the Dirac delta is mathematically elegant, care must be taken in physical interpretation:

- Real forces have finite duration and magnitude — delta models are idealizations.
- In numerical simulations, delta-like inputs can cause **instabilities** if not handled with proper approximation.
- For experimental data, **impulses are always spread in time and space**, so delta models are used to understand the **limiting behavior**.
