Chapter 9: Impulse and Response to Unit Impulse

Introduction

In the field of Earthquake Engineering, understanding the response of structures to impulsive forces is essential for predicting how buildings, bridges, and other infrastructure will behave during sudden excitations such as earthquakes. One of the most fundamental tools in analyzing such responses is the concept of **impulse** and the corresponding **response of systems to unit impulse**. This forms the basis for the **impulse response function**, a critical element in dynamic analysis and structural vibration studies. This chapter focuses on defining the nature of impulse forces, the characteristics of the unit impulse function (Dirac delta), and the way linear time-invariant (LTI) systems respond to such inputs.

9.1 Impulse Force and its Mathematical Representation

An **impulse** is defined as a force of very large magnitude acting over a very short period of time. Theoretically, this is modeled by the **Dirac delta function**, denoted as $\delta(t)$. The mathematical impulse I applied over a time interval t_1 to t_2 is:

$$I = \int_{t_1}^{t_2} F(t)\,dt$$

For a **unit impulse**, this simplifies to:

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Properties of Dirac Delta Function:

1. $\delta(t) = 0$ for all $t \neq 0$
2. $\int_{-\infty}^{\infty} \delta(t) dt = 1$
3. $\int_{-\infty}^{\infty} f(t)\delta(t - t_0) dt = f(t_0)$, known as the **sifting property**

Impulse functions allow us to test the behavior of dynamic systems at an instant, revealing core characteristics of the system response.

9.2 Equation of Motion for Single Degree of Freedom (SDOF) System

For a linear SDOF system subjected to a force $F(t)$, the equation of motion is:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

Where:

- m: Mass
- c: Damping coefficient
- k: Stiffness
- $x(t)$: Displacement
- $F(t)$: External force (here, impulse)

For unit impulse input: $F(t) = \delta(t)$

So, the governing equation becomes:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = \delta(t)$$

Solving this equation gives the **impulse response function** of the system.

9.3 Free Vibration Response of Undamped SDOF System to Unit Impulse

When damping $c = 0$, the equation simplifies to:

$$m\ddot{x}(t) + kx(t) = \delta(t)$$

Let $\omega_n = \sqrt{\frac{k}{m}}$ be the natural frequency.

Initial conditions just before impulse:

- $x(0^-) = 0$
- $\dot{x}(0^-) = 0$

Integrating the equation over a small interval around $t = 0$:

$$\int_{0^-}^{0^+} m\ddot{x}(t)dt + \int_{0^-}^{0^+} kx(t)dt = \int_{0^-}^{0^+} \delta(t)dt$$

$$m[\dot{x}(0^+) - \dot{x}(0^-)] = 1 \Rightarrow \dot{x}(0^+) = \frac{1}{m}$$

Then, the free vibration response for $t > 0$:

$$x(t) = \frac{1}{m\omega_n} \sin(\omega_n t)$$

This is the **impulse response** of an undamped SDOF system.

9.4 Response of Damped SDOF System to Unit Impulse

For a damped system, the equation of motion is:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = \delta(t)$$

Let:

- $\zeta = \frac{c}{2\sqrt{km}}$ be the damping ratio
- $\omega_n = \sqrt{\frac{k}{m}}$, natural frequency
- $\omega_d = \omega_n \sqrt{1 - \zeta^2}$, damped natural frequency

Assuming initial conditions:

- $x(0^-) = 0$, $\dot{x}(0^-) = 0$
- $\dot{x}(0^+) = \frac{1}{m}$

Then for $\zeta < 1$ (underdamped case):

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t), \quad t > 0$$

This describes how the response decays over time due to damping.

9.5 Unit Impulse Response Function (Green's Function)

The **impulse response function**, also called **Green's function** $h(t)$, is defined as the response of a system to a unit impulse input $\delta(t)$.

For a linear time-invariant system, the total response to any arbitrary force $F(t)$ can be obtained using **convolution**:

$$x(t) = \int_0^t h(t - \tau) F(\tau)\, d\tau$$

Thus, knowing $h(t)$ allows us to determine system response to any general input.

9.6 Physical Interpretation in Earthquake Engineering

In the context of Earthquake Engineering:

- Ground accelerations can be approximated as a sequence of impulse-like forces.
- The impulse response function helps in constructing the response history of the structure to seismic excitations.
- Structural control and damping devices are designed based on knowledge of how structures respond to impulsive disturbances.

9.7 Numerical Example

Given:

- $m = 1\,\text{kg}, \quad c = 1\,\text{N}\cdot\text{s/m}, \quad k = 4\,\text{N/m}$

Then:

- $\omega_n = 2\,\text{rad/s}$, $\zeta = \frac{1}{2\sqrt{4\cdot 1}} = 0.25$
- $\omega_d = 2\sqrt{1 - 0.25^2} = 1.936\,\text{rad/s}$

Impulse response:

$$x(t) = \frac{1}{1 \cdot 1.936} e^{-0.25\cdot 2\cdot t} \sin(1.936t)$$

$$x(t) = 0.5166 \cdot e^{-0.5t} \sin(1.936t)$$

This function describes the displacement response of the structure to an impulsive force at $t = 0$.

9.8 Applications in System Identification and Seismic Analysis

Impulse response functions are used in:

1. **System identification** – determining system parameters from measured response data.
2. **Finite Element Analysis (FEA)** – modal superposition methods use impulse response to construct time-history response.
3. **Base isolation systems** – impulse testing helps evaluate the performance of damping and isolating devices.

9.9 Convolution Integral and General Force Response

When a system is subjected to a general external force $F(t)$, rather than just an impulse, the response can be computed using the **convolution integral**. The convolution of the impulse response function $h(t)$ with the applied force gives the total response:

$$x(t) = \int_0^t h(t - \tau)F(\tau)\, d\tau$$

This equation implies that the total response is a superposition of the system's responses to infinitesimal impulse forces applied at each time instant.

Key Points:

- Applicable only for linear time-invariant (LTI) systems.
- It helps in solving systems with arbitrary input forces like earthquake ground motions.
- In structural dynamics, it allows time-history analysis when direct integration methods are complex.

9.10 Graphical Interpretation of Impulse Response

The **impulse response function** can be visualized to understand physical behavior:

- **Undamped System**: The response is a sine wave starting from zero with initial velocity.
- **Damped System**: The response shows decaying oscillations. The decay rate depends on the damping ratio ζ.
- **Critically Damped**: System returns to rest without oscillating.
- **Overdamped**: Slow return to equilibrium without overshooting.

Graphs Typically Include:

- Displacement vs. Time
- Velocity vs. Time
- Influence of damping on amplitude and duration

These plots are especially important in earthquake-resistant design, where the designer must control amplitude and duration of vibration.

9.11 Frequency Domain Representation of Impulse Response

Impulse response in time domain can be transformed into frequency domain using **Fourier Transform**:

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt$$

This frequency response function $H(\omega)$ characterizes how the system responds to sinusoidal inputs of different frequencies.

Applications in Earthquake Engineering:

- Spectral analysis of ground motion
- Frequency-based filtering of input signals
- Identifying natural frequencies and resonant behavior of structures

---

9.12 Importance in Seismic Design and Analysis

Impulse response analysis has critical applications in:

1. **Time-history analysis** of structural response under earthquake records.
2. **Seismic hazard modeling** – many earthquake records are modeled as impulse sequences.
3. **Response spectrum method** – derived from impulse responses of SDOF systems with varying natural frequencies.
4. **Design of base-isolated and damped systems** – impulse testing is used to evaluate system performance.

By understanding the impulse response, civil engineers can predict how structures behave under short, sudden inputs that mimic real earthquake forces.

9.13 Experimental Determination of Impulse Response

Impulse response can also be measured experimentally using:

- **Impact hammer tests** on structures.
- **Shaker systems** applying a known pulse.
- **Laser vibrometry** to record system displacement or acceleration.

From these tests, the impulse response function is extracted and used to calibrate computational models or validate design assumptions.

9.14 Extension to Multi-Degree-of-Freedom (MDOF) Systems

For **MDOF systems**, the equation of motion becomes:

$$[M]\{\ddot{x}(t)\} + [C]\{\dot{x}(t)\} + [K]\{x(t)\} = \{F(t)\}$$

The **impulse response function** is now a **matrix** $H(t)$, which relates each degree of freedom's response to an impulse applied at each DOF:

$$\{x(t)\} = \int_0^t H(t-\tau)\{F(\tau)\}d\tau$$

Modal analysis can be used to compute $H(t)$ using eigenvectors and eigenvalues, reducing a complex system to several SDOFs.

9.15 Computational Approaches

Due to complexity, most real-world impulse response problems are solved using **numerical techniques**:

- **Newmark-beta method**
- **Runge-Kutta methods**
- **State-space approaches**
- **Direct convolution methods using digital signal processing (DSP)**

Software like MATLAB, SAP2000, and ETABS allows engineers to simulate impulse and general dynamic response of structures efficiently.
