Chapter 3: Second-Order Homogeneous Equations with Constant Coefficients

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Introduction

Second-order differential equations frequently appear in the field of civil engineering, especially in structural analysis, fluid dynamics, soil mechanics, and vibration problems. A particularly important class is the **homogeneous linear differential equations with constant coefficients**, which allows for analytical solutions using exponential functions. This chapter provides a detailed treatment of the general form of these equations, methods of solving them, and examples relevant to engineering contexts.

3.1 General Form of the Equation

A **second-order homogeneous linear differential equation with constant coefficients** takes the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Where:

- a, b, c are constants (real numbers),
- $y(x)$ is the unknown function,
- $\frac{d^2y}{dx^2}$ is the second derivative,
- The equation is **homogeneous** because the right-hand side is zero.

This type of equation models many real-world phenomena, such as:

- Vibrations in beams (Euler-Bernoulli beam theory),
- Free oscillations of structures,
- Groundwater flow under steady-state conditions.

3.2 Characteristic Equation

To solve the differential equation, we assume a solution of the form:

$$y = e^{rx}$$

Substituting into the original equation:

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

Divide through by e^{rx} (never zero):

$$ar^2 + br + c = 0$$

This is known as the **characteristic equation** or **auxiliary equation**.

$$ar^2 + br + c = 0$$

Solve this quadratic to find roots r_1 and r_2. The nature of the roots determines the general solution.

3.3 Cases Based on Nature of Roots

Case 1: Distinct Real Roots ($D = b^2 - 4ac > 0$) Let the roots be r_1 and r_2, with $r_1 \neq r_2$ and both real.

General Solution:

$$y(x) = C_1e^{r_1x} + C_2e^{r_2x}$$

Where C_1 and C_2 are arbitrary constants determined by initial or boundary conditions.

Example:

$$y'' - 5y' + 6y = 0 \Rightarrow r^2 - 5r + 6 = 0 \Rightarrow r = 2, 3$$

$$\Rightarrow y(x) = C_1e^{2x} + C_2e^{3x}$$

Case 2: Repeated Real Roots ($D = 0$) Let the root be $r_1 = r_2 = r$.

General Solution:

$$y(x) = (C_1 + C_2x)e^{rx}$$

Example:

$$y'' - 4y' + 4y = 0 \Rightarrow r^2 - 4r + 4 = 0 \Rightarrow r = 2$$

$$\Rightarrow y(x) = (C_1 + C_2 x)e^{2x}$$

Case 3: Complex Roots ($D < 0$) Let the roots be complex: $r = \alpha \pm i\beta$

General Solution:

$$y(x) = e^{\alpha x} \left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right)$$

This represents damped oscillations—highly relevant in civil engineering (e.g., vibration analysis, seismic behavior).

Example:

$$y'' + 2y' + 5y = 0 \Rightarrow r^2 + 2r + 5 = 0 \Rightarrow r = -1 \pm 2i$$

$$\Rightarrow y(x) = e^{-x} \left(C_1 \cos(2x) + C_2 \sin(2x)\right)$$

3.4 Applications in Civil Engineering

1. Free Vibration of Structures In modeling free vibrations of a mass-spring system or cantilever beam:

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

Where:

- m: mass
- c: damping coefficient
- k: stiffness

This is a second-order homogeneous ODE. Its solution tells us whether the structure oscillates, settles, or diverges over time.

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2. Deflection of Beams (Euler-Bernoulli Beam Theory) Governing equation:

$$EI\frac{d^4y}{dx^4} = q(x)$$

For constant load $q(x) = 0$, this reduces to:

$$\frac{d^4y}{dx^4} = 0 \Rightarrow \text{Integrating twice leads to second-order ODEs.}$$

These can often be handled using the methods in this chapter.

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3. Groundwater Flow (Steady-State) Laplace's equation in 1D steady-state flow:

$$\frac{d^2h}{dx^2} = 0$$

Again, this is a homogeneous second-order equation with real constant coefficients.

3.5 Initial and Boundary Conditions

To obtain a **unique solution**, we often need initial or boundary values.

For example:

$$y(0) = y_0, \quad y'(0) = y_1$$

Substitute these into the general solution to determine C_1 and C_2.

3.6 Methodical Approach to Solving Second-Order Homogeneous Equations

Solving these equations systematically helps in mastering the concept:

Step 1: Write the Differential Equation Ensure it's in the standard form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Step 2: Form the Characteristic Equation

$$ar^2 + br + c = 0$$

Step 3: Find the Roots Use the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Step 4: Determine the Type of Roots

- Real and distinct
- Real and equal
- Complex conjugate

Step 5: Write the General Solution Based on Root Type Refer to Section 3.3 for the structure of the solution.

Step 6: Apply Initial/Boundary Conditions Use given values of $y(x_0)$ and $y'(x_0)$ to solve for constants C_1, C_2.

3.7 Solved Examples

Example 1: Real and Distinct Roots Solve:

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0, \quad y(0) = 3, \quad y'(0) = 5$$

Step 1: Characteristic equation:

$$r^2 - 7r + 10 = 0 \Rightarrow r = 2, 5$$

General solution:

$$y(x) = C_1 e^{2x} + C_2 e^{5x}$$

Apply initial conditions:

$$y(0) = C_1 + C_2 = 3 \quad \text{(i)}$$

$$y'(x) = 2C_1 e^{2x} + 5C_2 e^{5x} \Rightarrow y'(0) = 2C_1 + 5C_2 = 5 \quad \text{(ii)}$$

Solve equations (i) and (ii):

From (i): $C_1 = 3 - C_2$ Substitute into (ii):

$$2(3 - C_2) + 5C_2 = 5 \Rightarrow 6 - 2C_2 + 5C_2 = 5 \Rightarrow 3C_2 = -1 \Rightarrow C_2 = -\frac{1}{3}$$

Then $C_1 = 3 + \frac{1}{3} = \frac{10}{3}$

Final solution:

$$y(x) = \frac{10}{3}e^{2x} - \frac{1}{3}e^{5x}$$

Example 2: Repeated Roots Solve:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1$$

Characteristic equation:

$$r^2 - 4r + 4 = 0 \Rightarrow r = 2 \text{ (repeated)}$$

General solution:

$$y(x) = (C_1 + C_2 x)e^{2x}$$

Apply initial conditions:

$$y(0) = C_1 = 2 \quad \text{(i)}$$

$$y'(x) = [C_2 + 2(C_1 + C_2 x)]e^{2x} \Rightarrow y'(0) = (C_2 + 2C_1) = -1 \quad \text{(ii)}$$

From (i): $C_1 = 2$

Substitute into (ii): $C_2 + 4 = -1 \Rightarrow C_2 = -5$

Final solution:

$$y(x) = (2 - 5x)e^{2x}$$

3.8 Graphical Interpretation of Solutions

- **Real distinct roots** → Exponential growth/decay (no oscillation)
- **Repeated roots** → Exponential decay/growth with polynomial weight
- **Complex roots** → Oscillatory motion with damping/growth (spiral-like in phase space)

Include plots using software (MATLAB/Python) for:

- Overdamped systems (real roots)
- Critically damped (repeated roots)
- Underdamped systems (complex roots)

3.9 Engineering Insight: Damping in Vibrations

In structural dynamics, the equation:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Models a damped vibrating system. Define damping ratio:

$$\zeta = \frac{c}{2\sqrt{mk}}, \quad \omega_n = \sqrt{\frac{k}{m}}$$

Then the system's behavior is:

- $\zeta > 1$: Overdamped (real distinct roots)
- $\zeta = 1$: Critically damped (repeated roots)
- $\zeta < 1$: Underdamped (complex roots)

Application: Civil engineers design buildings and bridges to respond within a controlled damping range to seismic activity.

3.10 Problems for Practice

1. Solve:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

2. Solve and classify the roots:

$$y'' + y' + y = 0$$

3. A cantilever beam's deflection satisfies:

$$EI\frac{d^4y}{dx^4} = 0$$

- Integrate twice and show that the solution involves a second-order homogeneous ODE. Then solve assuming zero shear and moment at the free end.

4. Solve:

$$y'' - 6y' + 13y = 0, \quad y(0) = 0, \quad y'(0) = 2$$

Summary

- Second-order homogeneous equations with constant coefficients are foundational in modeling real-world engineering systems.
- The solution depends entirely on the **discriminant** $D = b^2 - 4ac$:
  - $D > 0$: Real distinct roots
  - $D = 0$: Real repeated root
  - $D < 0$: Complex conjugate roots
- Methods learned here apply directly to vibration analysis, structural deflection, and flow through porous media.
- The exponential function-based solutions allow for a wide range of physical behaviors (e.g., exponential decay, oscillations, over-damping/under-damping).