Solid Mechanics
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Lecture - 6
Balance of Angular Momentum (Contd.)

Welcome to Lecture 6! We will continue with our Angular Momentum Balance derivation.

1 Angular Momentum Balance (start time: 00:21)

1.1 Traction contribution (start time: 00:27)

In the last lecture we had derived the Torque due to traction forces as:

$$\underline{T}^{\text{traction}}_{/\underline{x}} = \sum_{i=1}^{3} (\underline{e}_i \times \underline{\underline{\sigma}}\, \underline{e}_i)\Delta V + o(\Delta V) \quad (1)$$

1.2 Body force contribution (start time: 00:35)

Similarly, we had derived the torque due to body force as:

$$\underline{T}^{\text{body force}}_{/\underline{x}} = o(\Delta V) \quad (2)$$

1.3 Dynamics term (start time: 00:41)

We need to now derive $\frac{d}{dt}(\underline{H}_{/\underline{x}})$. In Figure 1, $\underline{x}$ is the center of the cuboid as considered previously and $\underline{y}$ is any arbitrary point in the cuboid. If we consider a particle of mass m moving with velocity $\underline{v}$, its angular momentum is given by:

$$\vec{r}_{/O} \times m\underline{v} \quad (3)$$

Here, $\vec{r}_{/O}$ is the position vector of the particle relative to the point 'O' about which we are measuring the angular momentum. To find the angular momentum of the mass m in the cuboid about its center, we need to integrate over all the particles that it contains.

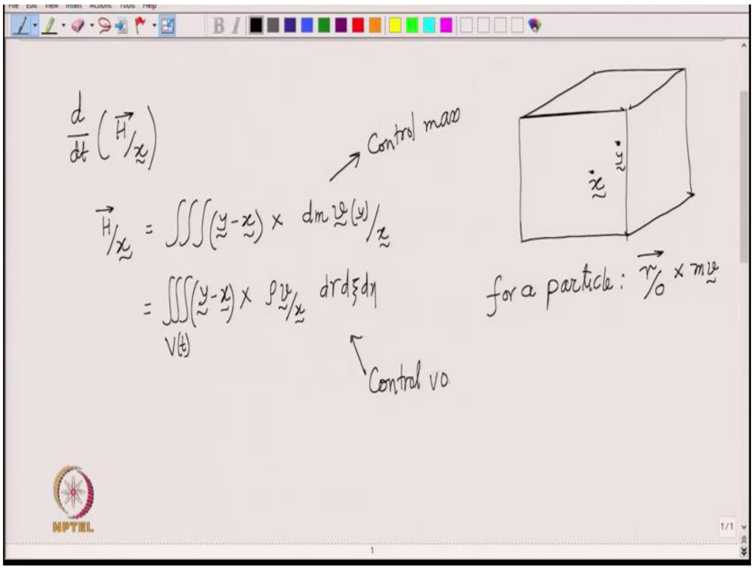

Figure 1: A cuboidal volume centered at the point of interest $\underline{x}$. $\underline{y}$ is an arbitrary point in the volume of the cuboid

As we are writing the equation in the center of mass frame, so the momentum of the particle is simply mass *dm* times the velocity of the particle $\underline{v}(\underline{y})$ relative to the center of mass, i.e., $\underline{v}(\underline{y})_{/\underline{x}}$. This leads to

$$\vec{H}_{/\underline{x}} = \iiint (\underline{y} - \underline{x}) \times dm\, \underline{v}(\underline{y})_{/\underline{x}} = \iiint_{V(t)} (\underline{y} - \underline{x}) \times \rho \underline{v}_{/\underline{x}}\, d\gamma d\xi d\eta \tag{4}$$

Note again that the first integral is over fixed/identifiable mass whereas the second integral is over changing volume domain since as the identifiable mass moves in space, the volume contained by it changes with time. This volume happens to be our cuboid in the current time. Just like linear momentum balance, angular momentum balance is also applied always to a fixed/identifiable mass. So, taking the time derivative in control volume setting isn't easy whereas in control mass setting, we can easily move the time derivative within the integral. Thus, taking the time derivative in control mass setting, we get

$$\frac{d}{dt}(\underline{H}_{/\underline{x}}) = \iiint_{mass} \underbrace{\frac{d}{dt}(\underline{y} - \underline{x})}_{\underline{v}(\underline{y})_{/\underline{x}}} \times dm\, \underline{v}(\underline{y})_{/\underline{x}} + \iiint_{mass} (\underline{y} - \underline{x}) \times dm\, \underline{a}(\underline{y})_{/\underline{x}} \tag{5}$$

Here, the time derivative of velocity becomes acceleration. In the first term, the time derivative of ($\underline{y} - \underline{x}$) will give velocity of $\underline{y}$ minus velocity of $\underline{x}$, i.e., $\underline{v}(\underline{y})_{/\underline{x}}$. So, this term cancels as it involves the cross product of a quantity with itself. Also note that *dm*, mass of the particle, is a constant and does not change with time: density and volume of a particle can change but mass cannot. For the second term, we can now switch back to the volume integral as the time derivative is already inside, i.e.,

$$\frac{d}{dt}(\underline{H}_{/\underline{x}}) = \iiint_{V} (\underline{y} - \underline{x}) \times \rho \underline{a}(\underline{y})_{/\underline{x}} d\gamma d\xi d\eta \tag{6}$$

Comparing this with the derivation of the torque due to body force term, we see that this is exactly similar to that term with just $\underline{b}$ replaced with $\rho \underline{a}(\underline{y})_{/\underline{x}}$. So, upon integrating this dynamic term using

Taylor's expansion for acceleration, we would get a similar result as that of the body force contribution to torque, i.e.,

$$\frac{d}{dt}(\underline{H}_{/\underline{x}}) = o(\Delta V) \tag{7}$$

1.4 Final balance (start time: 10:54)

So, now we can substitute equations (1),(2) and (7) in

$$\sum \underline{T}^{ext}_{/\underline{x}} = \frac{d}{dt}(\underline{H}_{/\underline{x}}) \tag{8}$$

$$\Rightarrow \sum_{i=1}^{3}(\underline{e}_i \times \underline{\underline{\sigma}}\, \underline{e}_i)\Delta V + o(\Delta V) = o(\Delta V) \tag{9}$$

As this equation is valid for a cuboid of any size, we can shrink it to its centroid ($\underline{x}$). So, first we divide by ΔV on both sides and then take the $\lim_{\Delta V \to 0}$:

$$\lim_{\Delta V \to 0}\left[\frac{\sum_{i=1}^{3}(\underline{e}_i \times \underline{\underline{\sigma}}\, \underline{e}_i)\Delta V + o(\Delta V)}{\Delta V} = \frac{o(\Delta V)}{\Delta V}\right]. \tag{10}$$

We finally get

$$\boxed{\sum_{i=1}^{3} \underline{e}_i \times \underline{\underline{\sigma}}\, \underline{e}_i = \underline{0}} \quad \rightarrow \textbf{Angular Momentum Balance (AMB)} \tag{11}$$

This equation holds even if a body force is acting or if the body is accelerating as these terms vanished in the derivation itself.

1.5 Representation in a coordinate system (start time: 14:25)

Let us try to write the tensor equation (11) in component form in ($\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$) coordinate system. Consider the first term in the summation. The representation of $\underline{e}_1$ in ($\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$) coordinate system is trivial and is given as $[1\ 0\ 0]^T$ while the representation of $\underline{\underline{\sigma}}\ \underline{e}_1$ will just be the first column of the stress matrix corresponding to ($\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$) coordinate system. We thus have

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \times \begin{bmatrix}\sigma_{11}\\ \tau_{21}\\ \tau_{31}\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} \times \begin{bmatrix}\tau_{12}\\ \sigma_{22}\\ \tau_{32}\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}\tau_{13}\\ \tau_{23}\\ \sigma_{33}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}0\\ -\tau_{31}\\ \tau_{21}\end{bmatrix} + \begin{bmatrix}\tau_{32}\\ 0\\ -\tau_{12}\end{bmatrix} + \begin{bmatrix}-\tau_{23}\\ \tau_{13}\\ 0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \tag{12}$$

Thus, we get the following three scalar equations:

$$\tau_{32} = \tau_{23}$$
$$\tau_{31} = \tau_{13}$$
$$\tau_{21} = \tau_{12} \qquad (13)$$

So, the final outcome of the angular momentum balance is that the stress matrix is symmetric which holds true even if the body is accelerating or body force is present.

1.6 An alternate method to derive AMB (start time: 19:02)

There is also a simpler but approximate way to come to this final outcome which is given in many textbooks. Consider the same cuboid again. The coordinate system ($\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$) is shown on the right side of Figure 2. We have drawn stress matrix components on some of the planes as shown in Figure 2. On the right plane, we have σ_{11}, τ_{21} and τ_{31}. On the top plane, we have σ_{22}, τ_{12} and τ_{32}. On the bottom face, we have σ_{22}, τ_{12} and τ_{32}. But, these three act in the $-\underline{e}_2$, $-\underline{e}_1$ and $-\underline{e}_3$ directions respectively. This is because this is the $-\underline{e}_2$ plane. Similarly, we can draw the components on the $-\underline{e}_1$ plane.

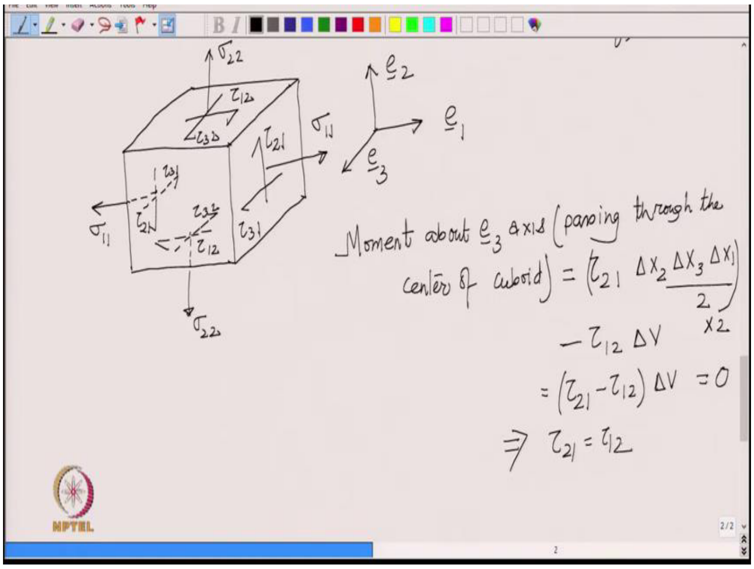

Figure 2: Traction components on various faces of the cuboid along with the coordinate system

Let us now find the moment due to these tractions about $\underline{e}_3$ axis passing through the centroid of the cuboid. The normal components σ_{11}, σ_{22} and σ_{33} pass through the centroid and thus do not contribute to this moment. The components τ_{31} and τ_{32} also do not contribute to this moment as they are parallel or anti-parallel to $\underline{e}_3$ direction. Only contributions would come from τ_{12} and τ_{21}. Moment due to τ_{21} will be traction times the area of the face ($\Delta x_2 \Delta x_3$) on which it acts times the distance from the center ($\frac{\Delta x_1}{2}$). The component τ_{21} has two contributions from $\underline{e}_1$ and $-\underline{e}_1$ planes both in the same direction ($+\underline{e}_3$). The moment due to two τ_{12} components acts in $-\underline{e}_3$ direction. Finally, moment due to body force will be a smaller order term. The rate of change of angular momentum about $\underline{e}_3$ axis is also a smaller order term. When we shrink the cuboid, their contribution will vanish quickly than the contribution to moment due to traction components. The total moment about $\underline{e}_3$ axis (passing through the center of the cuboid) will then be:

$$
\begin{aligned}
(\tau_{21}\frac{\Delta x_2 \Delta x_3 \Delta x_1}{2}) \times 2 - (\tau_{12}\frac{\Delta x_1 \Delta x_3 \Delta x_2}{2}) \times 2 &= 0 \\
(\tau_{21} - \tau_{12})\Delta V &= 0 \\
\Rightarrow \tau_{21} &= \tau_{12}
\end{aligned}
\quad (14)
$$

This is a simpler derivation but it involves a strong assumption of traction components not varying on cuboid's faces. Likewise, when we find the moment equation about $\underline{e}_1$ and $\underline{e}_2$ axis, we will get $\tau_{23} = \tau_{32}$ and $\tau_{13} = \tau_{31}$ respectively. The symmetry of stress matrix leads to several simplifications as we will see in future lectures.

2 Relating externally applied distributed load on body's surface to stress tensor(start time: 27:22)

Suppose we have an arbitrary body which is clamped at some part of the boundary and a load is applied on some part of the boundary by an external agent. As the load is usually distributed over an area of the boundary of the body, it has the unit of traction which we write as $\underline{t}_0$ as shown in Figure 3. Our aim is to relate the state of stress (at the point where external load is acting) to the applied load itself. If we solve the stress equilibrium equations, we can find stress everywhere in the body. But at points on the boundary where the external load acts, we can find the state of stress partially without solving the stress equilibrium equations. In fact, the relation that we will derive here will be required as boundary condition in order to fully solve the stress equilibrium equations.

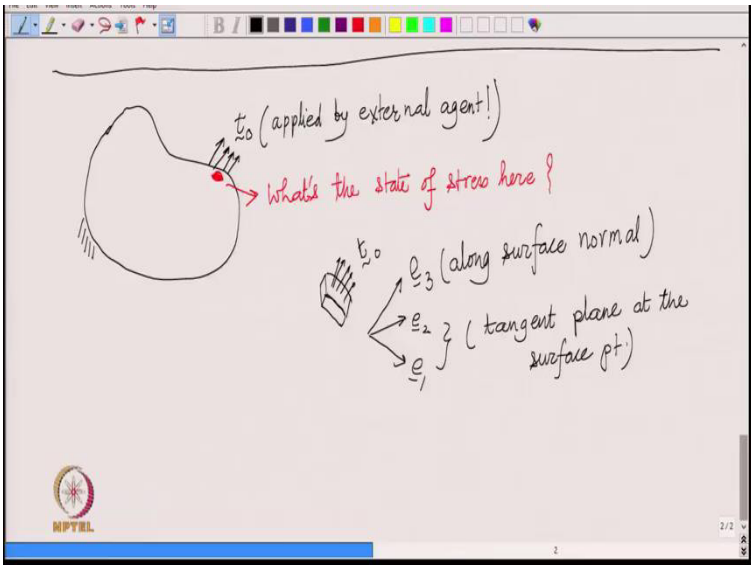

Figure 3: An arbitrary body subjected to an externally applied distributed load $\underline{t}_0$ over its surface

Let us set up our coordinate system such that we have $\underline{e}_3$ axis along the surface normal as shown in Figure 4. We have drawn the part of the body near the surface. On the top part of this surface, external load $\underline{t}_0$ acts. We are looking at an infinitesimal part of the body, so we can safely assume that $\underline{t}_0$ is constant on the top surface. The directions $\underline{e}_1$ and $\underline{e}_2$ are in the plane of the surface at that point and aligned along the edges of the small part of the body. We also say that the two directions are in the tangent plane at that point. So, $\underline{e}_3$ becomes the normal to this tangent plane. The edges along $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$ directions are of length Δ_1, Δ_2 and Δ_3 respectively.

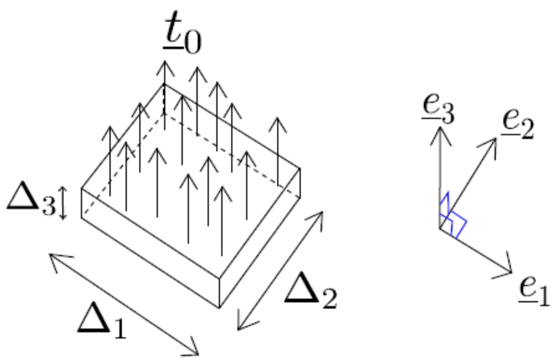

Figure 4: A zoomed view of the part of the body close to the surface point where the distributed external load acts

The total force on this small part of the body will be the vector sum of the traction forces applied from remaining part of the body, externally applied load and the body force, i.e.,

$$\underline{t}^{-3}\Delta_1\Delta_2 + (\underline{t}^2 + \underline{t}^{-2})\Delta_1\Delta_3 + (\underline{t}^1 + \underline{t}^{-1})\Delta_2\Delta_3 + \underline{b}\Delta_1\Delta_2\Delta_3 + \underline{t}_0\Delta_1\Delta_2$$

We now apply Newton's 2nd law to this small part of the body:

$$\underline{t}^{-3}\Delta_1\Delta_2 + (\underline{t}^2 + \underline{t}^{-2})\Delta_1\Delta_3 + (\underline{t}^1 + \underline{t}^{-1})\Delta_2\Delta_3 + \underline{b}\Delta_1\Delta_2\Delta_3 + \underline{t}_0\Delta_1\Delta_2 = \rho\Delta_1\Delta_2\Delta_3\underline{a} \tag{15}$$

This small box is also called a 'pill box' in several books. We now divide both sides of the above equation by $\Delta_1\Delta_2$ and then take $\lim_{\Delta_3\to 0}$. By taking this limit, we are essentially shrinking the pill box by pushing the bottom surface towards the top surface keeping the area of these surfaces ($\Delta_1\Delta_2$) constant.

$$\lim_{\Delta_3\to 0} \frac{\underline{t}^{-3}\Delta_1\Delta_2 + (\underline{t}^2 + \underline{t}^{-2})\Delta_1\Delta_3 + (\underline{t}^1 + \underline{t}^{-1})\Delta_2\Delta_3 + \underline{b}\Delta_1\Delta_2\Delta_3 + \underline{t}_0\Delta_1\Delta_2}{\Delta_1\Delta_2} = \frac{\rho\Delta_1\Delta_2\Delta_3\underline{a}}{\Delta_1\Delta_2}$$
$$\Rightarrow \lim_{\Delta_3\to 0}\left[\underline{t}^{-3} + \underline{t}_0 + (\underline{t}^2 + \underline{t}^{-2})\frac{\Delta_3}{\Delta_2} + (\underline{t}^1 + \underline{t}^{-1})\frac{\Delta_3}{\Delta_1} + (\underline{b} - \rho\underline{a})\Delta_3 = \underline{0}\right] \tag{16}$$

The terms containing Δ_3 will vanish in the limit and we get

$$\underline{t}^{-3} = -\underline{t}_0 \tag{17}$$

As $\underline{t}^3$ and $\underline{t}^{-3}$ form an action reaction pair, we finally obtain

$$\underline{t}^3 = \underline{\underline{\sigma}}\,\underline{e}_3 = \underline{t}_0 \tag{18}$$

As mentioned earlier, this equation is also used as boundary condition for solving the stress equilibrium equation. This result might look very intuitive and trivial. But, it is not as straightforward as it looks. Traction is the force per unit area applied by one part of the body on the other part of the body. It is applied by the body itself and is an internal force per unit area whereas $\underline{t}_0$ is applied by an external agent. We cannot conclude that $\underline{t}_0 = \underline{t}^3$ by intuition because they are two different things. But, this derivation tells us that the internal response of the body close to the surface is the same as what the external agent is applying. We still won't be able to conclude anything about $\underline{t}^1$ or $\underline{t}^2$ from this derivation though.

2.1 Relating stress matrix at surface point with externally applied load (start time: 40:40)

Suppose we want to find the stress matrix at the surface point with respect to the chosen coordinate system. We know that the columns of the stress matrix will be formed by the tractions on $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$ planes respectively. Thus, using equation (18), we know that the third column is the externally applied load per unit area. The first and second columns of the stress matrix are still unknown. As a stress matrix has to be symmetric, the third component of the first and second column also become known to us. So, we get five entries of the stress matrix at surface point right away as shown below:

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} \times & \times & \underline{t}_0^1 \\ \times & \times & \underline{t}_0^2 \\ \underline{t}_0^1 & \underline{t}_0^2 & \underline{t}_0^3 \end{bmatrix} \tag{19}$$

The rest four entries are unknown (out of which only three are independent as the stress matrix has to be symmetric) and can be found by solving the stress equilibrium equations. As a special case, if there are parts of the boundary where no external load is being applied ($\underline{t}_0=\underline{0}$), then

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{20}$$

3 Traction and Stress inside a fluid body at rest (start time: 43:56)

Think of a static fluid, say bucket filled with water as shown in Figure 5. We know that the pressure (p) inside the water is given by ρgh where h is the depth from the top surface, g is the acceleration due to gravity and ρ is the density of water.

3.1 Traction (start time: 44:49)

We want to know the state of stress at any point in the water body. So, we first think of a small infinitesimal plane at that point. As fluids cannot sustain shear when they are in static equilibrium, there will be no shear component of traction on any plane. The traction would be the same as pressure (p) and would act along the plane normal but pointing into the plane due to compressive nature of pressure. Thus, at an arbitrary point $\underline{x}$ in the fluid and on an arbitrary plane at that point (with plane normal given by $\underline{n}$), traction $\underline{t}$ will be given by

$$\underline{t}(\underline{x};\underline{n}) = -p(\underline{x})\ \underline{n} \tag{21}$$

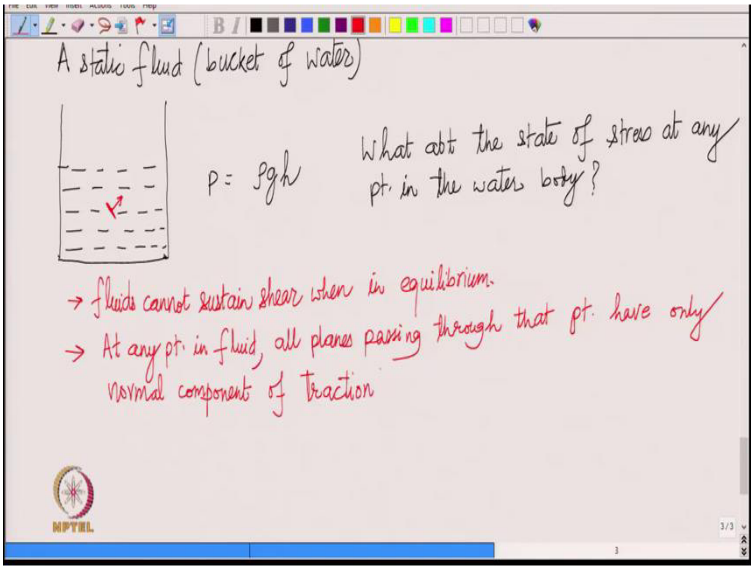

Figure 5: A bucket filled with water with a point $\underline{x}$ at depth h inside water.

3.2 Stress (start time: 48:37)

Stress matrix in ($\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$) coordinate system can be found by writing the tractions as columns. Traction on $\underline{e}_i$ plane will be equal to $-p\underline{e}_i$. So

$$\underline{\underline{\sigma}} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} = -p\,[\underline{\underline{I}}] \tag{22}$$

We observe that all the shear components in the stress matrix are zero. We can verify as shown below that $\underline{\underline{\sigma}}\,\underline{n}$ will give us the traction given by equation (21):

$$\underline{t} = \underline{\underline{\sigma}}\,\underline{n} = -p\underline{\underline{I}}\,\underline{n} = -p\underline{n} \tag{23}$$

Thus, the stress tensor for a fluid body in statics is $-p\underline{\underline{I}}$.