Chapter 7: Free Vibration of Single Degree of Freedom (SDOF) System

Introduction

The study of vibrations is a foundational concept in structural dynamics and earthquake engineering. A structure, when subjected to dynamic forces such as those generated by earthquakes, responds with motion that can be understood and predicted through vibration theory. One of the most basic and important systems in vibration analysis is the **Single Degree of Freedom (SDOF) system**. The understanding of free vibration of an SDOF system, in the absence of any external forces and damping, lays the groundwork for more complex dynamic analysis and is essential in seismic design and assessment of structures.

7.1 Definition of Free Vibration

Free vibration refers to the motion of a mechanical system when it is allowed to vibrate naturally without the influence of external forces after an initial disturbance. For a SDOF system, this typically involves a mass-spring system set into motion and allowed to oscillate freely.

7.2 Idealization of SDOF System

A single degree of freedom system is characterized by:

- **One mass** (m) which can move in only one direction.
- **One spring** with stiffness (k).
- No damping or external forcing (for the undamped case).
- A displacement function **x(t)** which fully describes the motion.

This system is often represented by a mass attached to a spring that can move vertically or horizontally, depending on the setup.

7.3 Equation of Motion for Free Vibration

The general equation of motion for an undamped SDOF system undergoing free vibration is:

$$m\ddot{x}(t)+kx(t)=0$$

Where:

- m = mass of the system
- k = stiffness of the spring
- $x(t)$ = displacement as a function of time
- $\ddot{x}(t)$ = acceleration (second derivative of displacement)

Dividing the equation by m, we get:

$$\ddot{x}(t)+\omega_n^2 x(t)=0$$

Where $\omega_n=\sqrt{\frac{k}{m}}$ is the **natural circular frequency** of the system.

7.4 Solution to the Equation of Motion

The solution to the second-order differential equation is of the form:

$$x(t)=A\cos(\omega_n t)+B\sin(\omega_n t)$$

Where:

- A and B are constants determined by initial conditions (initial displacement x_0 and initial velocity $\dot{x}_0$).

Alternatively, using the harmonic form:

$$x(t)=X\cos(\omega_n t+\phi)$$

Where:

- X = amplitude of vibration
- ϕ = phase angle

These parameters can be related back to the initial conditions using:

$$X=\sqrt{A^2+B^2},\tan(\phi)=\frac{B}{A}$$

7.5 Natural Frequency and Time Period

Natural Frequency (ω_n):

$$\omega_n = \sqrt{\frac{k}{m}} \text{(in radians/sec)}$$

Time Period (T):

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}} \text{(in seconds)}$$

The natural frequency and time period are intrinsic properties of the system and determine how fast or slow the system oscillates under free vibration.

7.6 Initial Conditions and System Response

Given:

- Initial displacement: $x(0) = x_0$
- Initial velocity: $\dot{x}(0) = v_0$

Then the constants in the general solution are:

$$A = x_0, B = \frac{v_0}{\omega_n}$$

Therefore:

$$x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t)$$

This form is particularly useful in practical applications when the initial state of the system is known.

7.7 Graphical Representation of Motion

The motion of an undamped SDOF system is sinusoidal and periodic. The displacement-time plot shows a smooth oscillation about the equilibrium position.

The frequency remains constant and amplitude does not decay over time (in the ideal undamped case).

Plots include:

- Displacement vs. time
- Velocity vs. time
- Acceleration vs. time
- Phase diagram (x vs. ẋ)

7.8 Energy in Free Vibration

In undamped free vibration, **total mechanical energy** is conserved:

- **Kinetic Energy (KE):**

$$KE = \frac{1}{2} m \dot{x}^2$$

- **Potential Energy (PE):**

$$PE = \frac{1}{2} k x^2$$

- **Total Energy (E):**

$$E = KE + PE = \frac{1}{2} k X^2 = \text{constant}$$

Energy keeps transferring between kinetic and potential forms during vibration.

7.9 Effect of Mass and Stiffness

- Increase in **mass (m)** → decreases natural frequency → system vibrates more slowly.
- Increase in **stiffness (k)** → increases natural frequency → system vibrates faster.

Understanding this relation is vital in designing structures to avoid resonance during earthquakes.

7.10 Resonance in SDOF Systems

Although resonance is more critical in forced vibration, understanding free vibration helps explain why a system vibrates violently when excited at its natural frequency. Avoiding this frequency is a key part of structural design in earthquake-prone zones.

7.11 Real-World Application in Earthquake Engineering

- Most structures can be idealized as SDOF systems for initial analysis.
- Helps predict how a structure will respond when an earthquake provides an initial displacement or velocity.
- Provides a benchmark for validating computational models and design codes.

7.12 Damped vs Undamped Free Vibration (Introduction to Damping)

Although this chapter primarily focuses on undamped systems, a brief understanding of damping is essential for transitioning to real-world systems.

- In undamped systems, oscillations continue indefinitely with constant amplitude.
- Real structures exhibit energy dissipation due to internal material friction, air resistance, and connection slippage.
- Introduction of damping modifies the differential equation:

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=0$$

Where c = damping coefficient.

The behavior of the system then depends on the **damping ratio**:

$$\zeta=\frac{c}{2\sqrt{km}}$$

This concept sets the stage for understanding **critical damping**, **underdamping**, and **overdamping**, which are central to earthquake-resistant design.

7.13 Logarithmic Decrement (δ)

Logarithmic decrement is used to measure damping in systems by evaluating the rate at which amplitude decays:

$$\delta = \frac{1}{n} \ln\left(\frac{x(t)}{x(t+nT)}\right)$$

Where:

- n = number of cycles
- T = time period
- $x(t)$, $x(t+nT)$ = amplitudes at different times

It is useful in experimental modal analysis for damping estimation of real structures.

7.14 Phase Plane Representation

Phase plane plots represent motion using velocity vs. displacement:

- The trajectory forms an **ellipse** in undamped systems.
- The shape and orientation help identify dynamic behavior visually.
- Useful in nonlinear systems and stability analysis.

7.15 Numerical Solution of Free Vibration (Finite Difference Method)

In practical problems where an analytical solution is not possible (e.g., complex geometries or variable properties), numerical methods are employed.

The **finite difference method** involves:

- Discretizing time
- Approximating derivatives
- Solving iteratively for $x(t)$

For example, using central difference:

$$x_{i+1} = 2x_i - x_{i-1} - \omega_n^2 x_i \Delta t^2$$

Where Δt is the time step.

7.16 Modal Parameters and Experimental Determination

Engineers often need to determine a structure's natural frequency and mode shapes experimentally.

- **Modal Testing** involves applying a known disturbance and measuring response.
- **FFT (Fast Fourier Transform)** and **Frequency Response Function (FRF)** are used to extract:
    - Natural frequency
    - Damping ratio
    - Mode shapes

These tests are often conducted in laboratories or in-field seismic evaluations.

7.17 Vibration Isolation and Structural Implications

Understanding free vibration behavior is essential in designing systems with **vibration isolation**:

- Structures must be isolated from ground motion to reduce amplification.
- Use of **base isolators**, **rubber bearings**, or **tuned mass dampers** is based on altering the natural frequency of the structure.

Design aims to avoid matching the building's natural frequency with dominant earthquake frequencies.

7.18 Role in Seismic Design Codes

Design codes such as **IS 1893**, **ASCE 7**, and **Eurocode 8** utilize fundamental natural periods (derived from SDOF behavior) to:

- Calculate base shear
- Assign importance factors

- Determine design spectra

The simplified SDOF model is the core assumption in **response spectrum analysis** used in most earthquake-resistant structural designs.

7.19 Case Studies and Real-World Observations

Examples from past earthquakes (e.g., Bhuj 2001, Nepal 2015) demonstrate how:

- Buildings with unfavorable natural frequencies failed due to resonance.
- Tall and flexible structures experienced higher modes.
- Retrofitting solutions were applied based on vibration characteristics.

Field studies validate the accuracy of SDOF assumptions in preliminary analysis.
