Chapter 18: Separation of Variables, Use of Fourier Series

Introduction

In Civil Engineering, the analysis of structures, heat conduction, fluid flow, and wave propagation often leads to solving **partial differential equations (PDEs)**. A powerful analytical technique for solving linear PDEs is the **Method of Separation of Variables** combined with the **Fourier Series** for expressing complex boundary or initial conditions.

This chapter provides an in-depth exploration of how separation of variables can be used to reduce a PDE into simpler ordinary differential equations (ODEs), and how Fourier series allow us to express complex functions as infinite sums of sines and cosines to meet boundary and initial conditions.

18.1 Partial Differential Equations (PDEs) and their Types

A **partial differential equation** involves partial derivatives of a multivariable function. The three classical types of second-order PDEs relevant to civil engineering problems are:

- **Elliptic PDEs**: e.g., Laplace's equation, for steady-state heat distribution or potential flow.
- **Parabolic PDEs**: e.g., Heat equation, for time-dependent heat conduction problems.
- **Hyperbolic PDEs**: e.g., Wave equation, for vibration of structures or wave propagation.

18.2 Method of Separation of Variables

The **Separation of Variables** method assumes the solution of a PDE can be written as the product of single-variable functions. This method is applicable to linear PDEs with homogeneous boundary conditions.

General Procedure:

1. **Assume** a solution of the form:

$$u(x,t) = X(x)T(t)$$

2. **Substitute** into the PDE.

3. **Separate** the equation such that each side depends on only one variable:

$$\frac{1}{T(t)}\frac{dT}{dt} = \frac{1}{X(x)}\frac{d^2X}{dx^2} = -\lambda$$

- where λ is the **separation constant**.

4. **Solve** the resulting ODEs:
   - One in x
   - One in t
5. **Apply boundary conditions** to determine allowed values of λ and corresponding eigenfunctions.
6. **Construct the general solution** as a series using the principle of superposition.

18.3 Example: Solving the One-Dimensional Heat Equation

The heat equation in one dimension is:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

with conditions:

- $u(0,t) = u(L,t) = 0$ (boundary conditions)
- $u(x,0) = f(x)$ (initial condition)

Step 1: Assume solution

$$u(x,t) = X(x)T(t)$$

Step 2: Substitute into PDE

$$X(x)\frac{dT}{dt} = \alpha^2 T(t)\frac{d^2X}{dx^2} \Rightarrow \frac{1}{\alpha^2 T(t)}\frac{dT}{dt} = \frac{1}{X(x)}\frac{d^2X}{dx^2} = -\lambda$$

Step 3: Solve ODEs

- **Temporal ODE**:

$$\frac{dT}{dt} + \alpha^2\lambda T = 0 \Rightarrow T(t) = Ce^{-\alpha^2\lambda t}$$

- **Spatial ODE**:

$$\frac{d^2X}{dx^2} + \lambda X = 0$$

- Applying boundary conditions $X(0) = X(L) = 0$ leads to:

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Step 4: General solution

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

18.4 Fourier Series

A **Fourier series** represents a periodic function $f(x)$ as a sum of sines and cosines.

18.4.1 Fourier Series on $[-L, L]$

For a piecewise continuous function $f(x)$ on $[-L, L]$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

Where coefficients are:

- $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$
- $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$
- $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$

18.4.2 Fourier Sine and Cosine Series

For functions defined on $[0, L]$, we can define:

- **Fourier sine series** (odd extension)
- **Fourier cosine series** (even extension)

These are used to match boundary conditions in PDE problems:

- Dirichlet BCs (zero at boundaries): Use **sine series**
- Neumann BCs (zero derivative at boundaries): Use **cosine series**

18.5 Application of Fourier Series in PDE Solutions

Once the eigenfunctions from the separation of variables are found (usually sine or cosine terms), the unknown coefficients C_n are determined using the **initial condition** via **Fourier expansion**:

$$C_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

This allows expressing any arbitrary initial state (e.g., temperature distribution) as a weighted sum of the eigenfunctions.

18.6 Application in Civil Engineering

- **Heat Transfer in Structures**: Determining temperature distribution in walls or slabs.
- **Vibration Analysis**: Solving wave equations in beams or plates.
- **Fluid Flow**: Solving Laplace's equation for velocity potential in flow through porous media.
- **Stress Analysis**: Distribution of stress and strain in elastic media under load.

18.7 Key Observations

- The **method of separation of variables** works well for linear PDEs with **homogeneous boundary conditions**.
- **Fourier series** provide the mathematical framework to express complex boundary/initial conditions.
- Eigenfunctions from spatial ODEs form an **orthogonal basis**, enabling series solutions.

18.8 Orthogonality and Eigenfunction Expansion

18.8.1 Orthogonality Property

The eigenfunctions $\{\sin\left(\frac{n\pi x}{L}\right)\}$ arising from Sturm–Liouville problems are **orthogonal** over the interval $[0, L]$:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \end{cases}$$

This orthogonality is crucial for computing Fourier coefficients and ensures the **uniqueness** and **completeness** of the series solution.

18.8.2 Fourier Coefficients from Inner Products

Let $\phi_n(x)$ be the nth eigenfunction. Then the projection of $f(x)$ onto $\phi_n(x)$ (using the inner product) gives:

$$C_n = \frac{\langle f(x), \phi_n(x) \rangle}{\langle \phi_n(x), \phi_n(x) \rangle}$$

This inner product viewpoint forms the basis of **modal analysis** in structural engineering.

18.9 Civil Engineering Example: Temperature in a Concrete Slab

Problem Statement

A concrete slab of length $L = 10$ m initially has a temperature distribution given by:

$$u(x, 0) = f(x) = 100 \sin \left(\frac{3\pi x}{10} \right)$$

with insulated boundaries ($u(0, t) = u(10, t) = 0$).

Using Separation of Variables:

We identify that:

- Only the $n = 3$ term is non-zero in Fourier sine expansion.
- Thus, the solution is:

$$u(x, t) = 100 \sin \left(\frac{3\pi x}{10} \right) e^{-\alpha^2 \left(\frac{3\pi}{10} \right)^2 t}$$

Interpretation:

- This represents a decaying temperature wave.
- As $t \to \infty$, $u(x, t) \to 0$ (steady state reached).
- The **mode shape** (spatial distribution) does not change; only amplitude decays.

18.10 Graphical Interpretation of Solutions

18.10.1 Mode Shapes

Each eigenfunction corresponds to a **vibrational mode** in structures:

- $n = 1$: Fundamental mode (single half-wave)
- $n = 2$: First overtone (one full wave)
- $n = 3$: Second overtone, etc.

This is analogous to a vibrating beam or bridge where each term in the Fourier expansion represents a different natural frequency.

18.10.2 Heat Equation Animation (Conceptual)

- At $t = 0$: Temperature is fully described by $f(x)$
- As t increases: Higher frequency components die out faster
- Eventually: Only low-frequency modes (longer wavelengths) dominate

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18.11 Numerical and Computational Aspects

In practice, exact analytical solutions may not be feasible for complex boundary geometries. However, the Fourier method remains foundational in **Finite Element** and **Finite Difference** methods.

Truncation and Approximation

- Use first few terms (e.g., 5 or 10) of Fourier series to approximate the solution.
- Accuracy improves with more terms, especially for smooth $f(x)$.

Error Estimation

The **Gibbs Phenomenon** occurs when approximating discontinuous functions — oscillations near discontinuities remain even with many terms.

18.12 Application in Beam Vibrations (Wave Equation)

Wave Equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Boundary Conditions: $u(0, t) = u(L, t) = 0$

Initial Conditions:

- Displacement: $u(x,0) = f(x)$
- Velocity: $u_t(x,0) = g(x)$

Solution using Separation of Variables + Fourier Series:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

- A_n, B_n are computed from $f(x)$, $g(x)$
- Models transverse vibration of simply supported beam

18.13 Use in Fluid Flow – Laplace's Equation

Laplace's Equation in 2D:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Boundary Conditions depend on geometry:

- Potential flow over dam, weir
- Flow under sheet piles

Separation of Variables works if boundary is rectangular: Assume $\phi(x,y) = X(x)Y(y)$, solve ODEs accordingly.

Fourier sine/cosine series then help satisfy conditions like:

- Specified head on boundary
- No-flow (Neumann) conditions on impermeable walls
