Chapter 11: Multiple Degree of Freedom (MDOF) System

Introduction

In real-world structures, motion due to earthquakes cannot be accurately modeled with Single Degree of Freedom (SDOF) systems alone. Most structures such as buildings, bridges, and towers possess multiple masses distributed in space and can vibrate in several modes simultaneously. These types of systems are best represented as **Multiple Degree of Freedom (MDOF)** systems.

An MDOF system has **more than one independent coordinate** required to describe its motion completely. Analyzing such systems is crucial to understanding the dynamic behavior of real-life structures under seismic loads. The chapter explores methods to derive equations of motion for MDOF systems, modal analysis, natural frequencies, mode shapes, and solutions using numerical methods.

11.1 Characteristics of MDOF Systems

- **Definition**: A mechanical or structural system that requires two or more independent coordinates (degrees of freedom) to describe its motion.
- **Examples**:
  - A shear building model with multiple floors.
  - Multi-span bridges.
  - Towers with mass concentrated at various levels.
- **Key Properties**:
  - Each DOF has an associated mass and stiffness.
  - Coupled differential equations govern motion.
  - System responds in multiple vibration modes.

11.2 Equations of Motion for Undamped MDOF System

Consider an n-DOF linear system with lumped masses and linear springs. The general form of the equations of motion without damping and external forces is:

$$[M]\{\ddot{u}(t)\}+[K]\{u(t)\}=\{0\}$$

Where:

- $[M]$ = Mass matrix $(n \times n)$
- $[K]$ = Stiffness matrix $(n \times n)$
- $\{u(t)\}$ = Displacement vector
- $\{\ddot{u}(t)\}$ = Acceleration vector

Key Concepts:

- Mass matrix is usually diagonal in lumped-mass systems.
- Stiffness matrix is symmetric and positive-definite.
- This leads to a system of n coupled second-order differential equations.

11.3 Mode Shapes and Natural Frequencies

To determine the natural behavior of the system, we assume a harmonic solution:

$$\{u(t)\}=\{\phi\}\sin(\omega t)$$

Substituting into the equation of motion:

$$([K]-\omega^2[M])\{\phi\}=\{0\}$$

This is a **generalized eigenvalue problem**, where:

- ω^2 are the **eigenvalues** (square of natural frequencies)
- $\{\phi\}$ are the corresponding **eigenvectors** (mode shapes)

Properties:

- There are n natural frequencies and n mode shapes.
- Mode shapes are orthogonal with respect to both [M] and [K].

11.4 Orthogonality of Mode Shapes

The eigenvectors (mode shapes) satisfy orthogonality properties:

1. **Mass Orthogonality**:

$$\{\phi_i\}^T[M]\{\phi_j\}=0 \text{ for } i \neq j$$

2. **Stiffness Orthogonality**:

$$\{\phi_i\}^T[K]\{\phi_j\}=0 \text{ for } i \neq j$$

These properties are used to **decouple** the equations of motion and simplify analysis.

11.5 Normalization of Mode Shapes

Mode shapes can be **normalized** in two common ways:

- **Mass normalization**:

$$\{\phi_i\}^T[M]\{\phi_i\}=1$$

- **Stiffness normalization**:

$$\{\phi_i\}^T[K]\{\phi_i\}=1$$

Normalization is useful for modal superposition and simplification in computations.

11.6 Modal Analysis of Undamped MDOF Systems

Using the orthogonality of mode shapes, the equations of motion can be **decoupled** by expressing displacement as a sum of modal contributions:

$$\{u(t)\}=\sum_{i=1}^{n}\phi_i q_i(t)$$

Where:

- ϕ_i: Mode shape
- $q_i(t)$: Generalized (modal) coordinate

Substituting into the original equations yields n uncoupled SDOF equations:

$$\ddot{q}_i(t)+\omega_i^2 q_i(t)=0$$

These can be solved independently, and the total response is obtained by summing all modal responses.

11.7 Equations of Motion for Damped MDOF Systems

When damping is present:

$$[M]\{\ddot{u}\}+[C]\{\dot{u}\}+[K]\{u\}=\{0\}$$

- $[C]$: Damping matrix

If damping is **classical (proportional)**, i.e., $[C]=\alpha[M]+\beta[K]$, then the modal equations remain uncoupled:

$$\ddot{q}_i(t)+2\zeta_i\omega_i\dot{q}_i(t)+\omega_i^2 q_i(t)=0$$

Where:

- ζ_i: Modal damping ratio
- These equations are identical to damped SDOF systems.

11.8 Response of MDOF Systems to Dynamic Loading

When subjected to external forces or ground acceleration (e.g., earthquakes), the equation becomes:

$$[M]\{\ddot{u}(t)\}+[C]\{\dot{u}(t)\}+[K]\{u(t)\}=\{f(t)\}$$

Or, for base excitation due to earthquake:

$$[M]\{\ddot{u}(t)\}+[C]\{\dot{u}(t)\}+[K]\{u(t)\}=-[M]\{\ddot{u}_g(t)\}$$

Where $\{\ddot{u}_g(t)\}$ is the ground acceleration.

Using **modal superposition**, the solution is:

$$\{u(t)\}=\sum_{i=1}^{n}\phi_i q_i(t)$$

Each modal coordinate $q_i(t)$ is found by solving the uncoupled modal equation with forcing.

11.9 Numerical Solution Techniques

For large systems or irregular structures, **closed-form modal analysis** is impractical. Numerical methods are used:

- **Finite Element Method (FEM)** to derive [M] and [K].
- **Matrix iteration (e.g., Power Method)** for finding dominant modes.
- **Newmark's Method** or **Wilson-θ Method** for time integration.
- **Modal truncation**: Use only first few dominant modes for efficient computation.

11.10 Modal Participation Factor and Effective Mass

In seismic analysis, not all modes contribute equally. Key concepts:

- **Modal Participation Factor** (Γ_i):

$$\Gamma_i = \frac{\{\phi_i\}^T [M] \{1\}}{\{\phi_i\}^T [M] \{\phi_i\}}$$

It represents how much each mode participates in response to uniform base excitation.

- **Effective Modal Mass**:

$$M_i^{eff} = \Gamma_i^2 \cdot \{\phi_i\}^T [M] \{\phi_i\}$$

The **cumulative effective mass** helps identify how many modes are needed to capture 90–95% of total mass participation.

11.11 Lumped Mass Matrix and Shear Building Model

For practical structural models like multi-storey buildings:

- Mass is concentrated at each floor.
- Stiffness is represented as inter-storey springs.
- Leads to a **tridiagonal stiffness matrix**.
- Simplifies the formulation and numerical solution.

This model is commonly used in seismic response studies.

11.12 Example Problems and Applications

Typical problems include:

- Determining natural frequencies and mode shapes for 2-DOF and 3-DOF systems.
- Computing seismic response using modal analysis.
- Estimating base shear and floor accelerations.
- Application in time-history and response spectrum analysis.

Applications:

- Earthquake-resistant design.
- Structural health monitoring.
- Retrofitting and vibration control.

11.13 Concept of Modal Superposition Method

The **Modal Superposition Method** allows for simplification of the dynamic response of MDOF systems by transforming the coupled system of differential equations into uncoupled modal equations.

Steps Involved:

1. **Eigenvalue Analysis**: Obtain natural frequencies ω_i and mode shapes ϕ_i.
2. **Modal Transformation**: Express physical displacements $\{u(t)\}$ as linear combination of mode shapes:

$$\{u(t)\} = \sum_{i=1}^{n} \phi_i q_i(t)$$

3. **Formulate Modal Equations**: Each equation is now:

$$\ddot{q}_i(t) + 2\zeta_i \omega_i \dot{q}_i(t) + \omega_i^2 q_i(t) = \Gamma_i \ddot{u}_g(t)$$

4. **Solve Modal Equations**: Using numerical methods or analytical techniques.
5. **Reconstruct Total Response**: By superimposing each modal response.

Advantages:

- Reduces computational effort.
- Captures dynamic behavior accurately using few dominant modes.

11.14 Modal Truncation and Convergence

In most seismic design problems, only a few modes contribute significantly. This leads to the concept of **modal truncation**:

- Truncation involves considering only the first few modes (say, 3 to 5).
- It is justified by analyzing **cumulative modal mass participation**.
- Ensure at least **90–95% of the total mass** is captured.

Modal Mass Participation Ratio:

$$\eta_i = \frac{M_i^{eff}}{M_{total}} \times 100\%$$

Convergence Check:

- Add modes progressively.
- Stop when the additional mode contributes negligible increase in effective mass.

11.15 Rayleigh's Method for Approximate Frequency

When only the fundamental (first) frequency is needed, **Rayleigh's method** offers an approximate and quick approach:

Rayleigh Quotient:

$$\omega_1^2 \approx \frac{\{u\}^T [K] \{u\}}{\{u\}^T [M] \{u\}}$$

Where $\{u\}$ is a trial shape function, typically based on static deflection under gravity.

Advantages:

- Useful for hand-calculations.
- Especially helpful for checking software results.

11.16 Time History Analysis of MDOF Systems

Time history analysis is used when ground motion acceleration records are available:

- Complete dynamic response is calculated by direct integration over time.
- Requires:
  - Ground acceleration $\ddot{u}_g(t)$
  - Damping matrix
  - Accurate time step

Numerical Integration Methods:

- **Newmark-beta Method**
- **Wilson-θ Method**
- **Runge-Kutta Methods**

Applications:

- Design of critical infrastructure (e.g., hospitals, bridges).
- Verification of dynamic characteristics in structural modeling.

11.17 Response Spectrum Method for MDOF Systems

The **Response Spectrum Method** provides a simplified way to estimate peak response due to earthquake loading.

Process:

1. Perform modal analysis to obtain mode shapes and frequencies.
2. For each mode, read peak displacement/acceleration from a given **design response spectrum**.
3. Compute modal responses using modal participation factor.
4. Combine modal responses using appropriate **combination rules**:
   - **SRSS**: Square Root of the Sum of the Squares
   - **CQC**: Complete Quadratic Combination (for closely spaced modes)

This method is widely used in seismic codes (IS 1893, ASCE 7, etc.) for practical design.

11.18 Base Shear Calculation in MDOF Systems

Base shear is the total lateral force induced at the base of a structure due to seismic activity.

Steps:

1. Compute modal responses using response spectrum method.
2. Compute shear force in each storey using modal contributions.
3. Total base shear is obtained by combining modal shear forces.

Importance:

- Used for seismic design and detailing.
- Ensures foundation and structural components are adequately sized.

11.19 Coupled Lateral-Torsional Vibrations

In irregular or asymmetric buildings, lateral vibrations can couple with **torsional modes**, leading to complex behavior.

Causes:

- Eccentricity between center of mass and center of stiffness.
- Plan irregularity or uneven mass distribution.

Effects:

- Torsional amplification of displacements.
- Larger demands on corner columns and beams.

Consideration:

- Must be modeled using full 3D MDOF analysis.
- Earthquake codes (e.g., IS 1893) provide guidelines for torsional irregularity.

11.20 Seismic Design Implications of MDOF Behavior

Understanding MDOF response helps ensure:

- Proper estimation of lateral drifts and floor accelerations.
- Realistic modeling of base shear and internal forces.
- Safer and cost-effective seismic design.

Designers use:

- Dominant mode-based design for regular buildings.
- Multi-mode or time history methods for complex structures.