Chapter 12: Two Degree of Freedom System

Introduction

In structural dynamics and earthquake engineering, most real-world structures respond in multiple modes during seismic events. While single degree of freedom (SDOF) systems provide essential understanding, they are often insufficient to capture the dynamic characteristics of complex structures like frames, bridges, and towers. A **two degree of freedom (2-DOF)** system serves as the next logical step in approximating multi-degree systems. This model helps in analyzing coupled modes of vibration, resonance, and modal participation. Understanding 2-DOF systems provides deeper insights into **modal analysis**, **mode shapes**, and **natural frequencies**, which are crucial for earthquake-resistant design.

12.1 Concept of Two Degree of Freedom System

A 2-DOF system is defined as a dynamic system that requires two independent coordinates to describe its motion completely. These systems typically consist of **two masses connected by springs and/or dampers**, each capable of independent translational or rotational movement.

Example Systems:

- Two-story shear building
- Two-mass torsional vibration system
- Rigid beam supported by two flexible supports

Let the displacements of the two masses be $x_1(t)$ and $x_2(t)$.

12.2 Free Vibration of Undamped 2-DOF Systems

For an undamped system with two masses m_1 and m_2, and stiffnesses k_1, k_2, and coupling stiffness k_{12}, the equations of motion are:

$$m_1\ddot{x}_1 + k_1x_1 + k_{12}(x_1 - x_2) = 0$$

$$m_2\ddot{x}_2 + k_2x_2 + k_{12}(x_2 - x_1) = 0$$

Matrix Form:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0$$

Where:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

12.3 Natural Frequencies and Mode Shapes

To solve the homogeneous system:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0$$

Assume harmonic motion:

$$\mathbf{x}(t) = \mathbf{\Phi} e^{i\omega t}$$

Substitute into the equation to obtain:

$$(-\omega^2 \mathbf{M} + \mathbf{K})\mathbf{\Phi} = 0$$

This is a standard **eigenvalue problem**:

$$\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0$$

Solving this gives two natural frequencies: ω_1, ω_2. The corresponding eigenvectors give the **mode shapes** ϕ_1, ϕ_2.

12.4 Orthogonality of Mode Shapes

If ϕ_1 and ϕ_2 are the normalized mode shapes, they satisfy the orthogonality condition:

$$\phi_i^T \mathbf{M} \phi_j = 0 \quad \text{for} \quad i \neq j$$

$$\phi_i^T \mathbf{K} \phi_j = 0 \quad \text{for} \quad i \neq j$$

Orthogonality simplifies modal analysis by **decoupling** the equations of motion when transformed into modal coordinates.

12.5 Forced Vibration and Modal Analysis

When subjected to an external force $\mathbf{F}(t)$, the system becomes:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}(t)$$

Using modal transformation:

$$\mathbf{x}(t) = \mathbf{\Phi}\mathbf{q}(t)$$

where $\mathbf{q}(t)$ are the **modal coordinates**, and $\mathbf{\Phi}$ is the mode shape matrix.

The transformed equation:

$$\ddot{\mathbf{q}} + \mathbf{\Omega}^2\mathbf{q} = \mathbf{\Phi}^T\mathbf{F}(t)$$

Where $\mathbf{\Omega}$ is the diagonal matrix of natural frequencies.

Each modal equation is **uncoupled**, allowing individual analysis.

12.6 Damped 2-DOF Systems

When damping is included, the equations of motion become:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0$$

Where $\mathbf{C}$ is the damping matrix. If damping is **proportional** (Rayleigh damping):

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$$

Modal transformation still results in decoupled equations with damped response.

12.7 Response to Earthquake Ground Motion

When the base is subjected to ground acceleration $\ddot{x}_g(t)$, the relative motion equations become:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = -\mathbf{M}\mathbf{r}\ddot{x}_g(t)$$

Where $\mathbf{r}$ is the influence vector (usually [1, 1]^T).

Using modal analysis:

$$\ddot{q}_i + 2\xi_i\omega_i\dot{q}_i + \omega_i^2 q_i = -\Gamma_i \ddot{x}_g(t)$$

Where Γ_i is the modal participation factor:

$$\Gamma_i = \frac{\phi_i^T \mathbf{M}\mathbf{r}}{\phi_i^T \mathbf{M}\phi_i}$$

The total response is the superposition of modal responses.

12.8 Coupled Lateral-Torsional Vibration in 2-DOF

In structures like bridges or buildings with eccentric mass or stiffness, **lateral and torsional vibrations** may couple, forming a 2-DOF system:

- One DOF for lateral translation
- One DOF for torsional rotation

Such coupled systems are crucial in **asymmetric buildings** where **eccentricity** between the center of mass and stiffness causes torsional amplification.

12.9 Numerical Example

Consider a 2-DOF system with:

- $m_1 = m_2 = 1000\,\text{kg}$
- $k_1 = k_2 = 20000\,\text{N/m}, k_{12} = 10000\,\text{N/m}$

Form the mass and stiffness matrices, solve the eigenvalue problem, find:

- ω_1, ω_2 (natural frequencies)
- Corresponding normalized mode shapes
- Use modal superposition for dynamic response

This exercise reinforces practical understanding of 2-DOF dynamics.

12.10 Importance in Earthquake Engineering

- **Multi-story structures** behave like MDOF systems. 2-DOF is a simple yet insightful model.
- Helps understand **modal interaction**, **resonance**, and **mode dominance**.
- Basis for **seismic analysis** in response spectrum and time history methods.

- Essential for design of tuned mass dampers (TMDs) and base isolation systems.

Sure! Continuing from **Section 12.10**, here are the remaining essential topics of the chapter, written in full detail and depth, just as you'd expect in a standard university textbook for **Earthquake Engineering**:

12.11 Mode Coupling and Beating Phenomenon

When the two natural frequencies of a 2-DOF system are close to each other and the system is excited by a force that contains components near both frequencies, **mode coupling** may occur. This leads to a **beating phenomenon**, where the energy continuously transfers between the two modes.

Beating Characteristics:

- Appears as a modulated vibration with alternating high and low amplitudes.
- Especially observed in free vibration when both modes are excited with similar amplitudes.
- Physically important in assessing **vibration fatigue** and **resonance amplification** in bridges and tall buildings.

Mathematically, if:

$$x(t) = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)$$

Then for $\omega_1 \approx \omega_2$, beating results in:

$$x(t) \approx 2A \cos\left(\frac{\omega_1 - \omega_2}{2} t\right) \cos\left(\frac{\omega_1 + \omega_2}{2} t\right)$$

12.12 Modal Superposition for Earthquake Analysis

In real-world applications, especially in seismic design, the **modal superposition method** allows solving the dynamic response of structures using:

1. Calculation of **modal responses** (SDOF-like response for each mode).
2. **Superposition** of modal responses to get total system behavior.

Steps:

1. Find mode shapes ϕ_i and frequencies ω_i.
2. Calculate modal participation factors Γ_i.

3. Solve for modal response $q_i(t)$ using the ground motion $\ddot{x}_g(t)$.
4. Compute total displacement:

$$x(t) = \sum_{i=1}^{2} \phi_i q_i(t)$$

This method significantly reduces computational effort compared to solving the full system directly.

12.13 Response Spectrum Analysis for 2-DOF Systems

The **response spectrum method** is a widely used tool in earthquake engineering for estimating the maximum response of structures subjected to ground motion. For 2-DOF systems:

- Each mode is treated independently.
- Peak modal responses are computed using the **design spectrum**.
- Modes are combined using techniques like:
  - **SRSS (Square Root of the Sum of Squares)**
  - **CQC (Complete Quadratic Combination)** – when modes are closely spaced.

Equation:

$$x_{\max} = \sum_{i=1}^{2} \Gamma_i \phi_i S_{a,i}$$

Where $S_{a,i}$ is the spectral acceleration for mode i.

12.14 Vibration Control using TMDs (Tuned Mass Dampers)

A **tuned mass damper** is often modeled as a 2-DOF system where one of the masses represents the structure and the second represents the damper.

Design Considerations:

- The damper's natural frequency is tuned to the dominant mode of the main structure.
- Proper tuning reduces the **peak displacement and acceleration** during seismic excitation.

- Widely used in skyscrapers, chimneys, and towers.

12.15 Practical Applications of 2-DOF Systems in Civil Engineering

Structure Type	Equivalent 2-DOF Model Description
Two-story RC frame	Masses at floor levels, springs as story stiffness
Bridge piers with top deck	Deck mass and pier flexibility as lumped parameters
Base isolated buildings	One DOF for superstructure, second for base movement
Asymmetric buildings	Translational + torsional DOF due to eccentricity

These simplified models help in understanding **dynamic characteristics**, **resonance risks**, and **retrofit strategies** for existing structures.

12.16 MATLAB/Computational Implementation

Most engineering analysis today involves computational tools. A typical MATLAB script for analyzing a 2-DOF system includes:

1. Defining $\mathbf{M}, \mathbf{K}$
2. Solving the eigenvalue problem
3. Plotting mode shapes and natural frequencies
4. Simulating response to harmonic or earthquake base excitation

```
% Example: 2-DOF system analysis
M = [1000 0; 0 1000];
K = [30000 -10000; -10000 30000];
[V, D] = eig(K, M);
omega = sqrt(diag(D));
phi = V;

% Mode shape plot
figure;
plot([0 phi(1,1)], [0 1], '-o'); hold on;
plot([0 phi(2,1)], [0 2], '-o');
title('Mode Shapes');
```

12.17 Limitations of 2-DOF Models

While useful, 2-DOF systems have their limitations:

- Cannot capture **higher-mode effects** in tall or flexible structures.
- Not suitable for **irregular geometry** or **nonlinear material behavior**.
- Over-simplification may lead to **underestimation of seismic demand**.

For such cases, **multi-degree of freedom (MDOF)** or **finite element models (FEM)** are necessary.
