Discrete Mathematics
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Lecture -49
Tutorial 8

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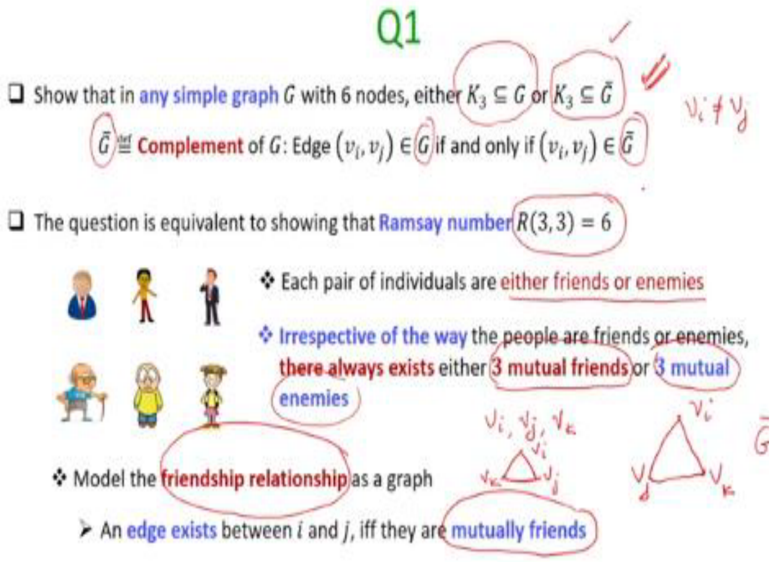

Hello everyone, welcome to tutorial number 8. So, we start with question number 1 where we want to prove that you take any simple graph G with 6 nodes, either the complete graph $K_3 \subseteq G$ or the complete graph $K_3 \subseteq \bar{G}$. So, let me first define what exactly is the complement of a graph in general.

So, the complement of a graph G is a graph which has the same vertex set as the vertex set G. And the edge set of $\bar{G}$ is complement of the edge set of the graph G, namely if you have an edge between the nodes v_i and v_j in the graph G, then the edge will not be present in the $\bar{G}$ and vice versa, where $v_i \neq v_j$. So, that is the general definition of a complement of a graph. And we want to prove this property in any simple graph with 6 nodes.

So, if you see closely then this question is equivalent to showing that the Ramsay number $R(3, 3)$ or the Ramsay function R(3, 3) is 6. Why so? So, recall what exactly are Ramsay numbers? So what we want to prove is that if you take any party where you have 6 guests and if each pair of distinct individuals are either friends or enemies. Then, we prove that irrespective of the way, the people are friends or enemies with each other, they are always exist either 3 mutual friends, namely we can find 3 friends F_1, F_2, F_3 who are mutually friends with each other.

That means F_1 and F_2 are mutual friends, F_2 and F_3 are mutual friends and F_1 and F_3 are mutual friends or we can always find 3 mutual enemies in the party. At least we can find 3 persons, person 1, person 2, person 3 such that p_1 and p_2 are not friends with each other, 2 and 3 are not friends with each other and 1 and 3 are also not friends with each other. So, we can model the friendship relation as an undirected graph, where I can say that there exists an edge between the person i and person j if and only if they are mutually friends.

If we do that then the friendship status of 6 people in a party, any party can be represented by a simple graph with 6 nodes and since we had proved that R(3, 3) = 6, that is equivalent to showing this property that we are supposed to prove. You wanted to prove that R(3, 3) is = 6 is equivalent to showing that if you take the friendship graph, you can always find 3 nodes in the friendship graph, such that among those 3 nodes you have edges, you can always find v_i, v_j and v_k such that you have the edges between v_i, v_j, you have the edge between v_j and v_k and you have the edge between v_k and $v_{i.}$ Or you have 3 people such that between i and j no edges there, between j and k no edge is there and between k and i no edges there in the graph G. If no edges are there among these three nodes in the graph G, then in the $\bar{G}$ graph, you will have an edge between the edge nodes v_i and v_j, the nodes v_j and v_k and the nodes v_k and v_i.

So, that is equivalent to showing that K3 is present or $K_3 \subseteq \bar{G}$ graph. So, this question we have already solved in principle. It is just that we are now getting a graph theoretic interpretation of the friendship relationship.

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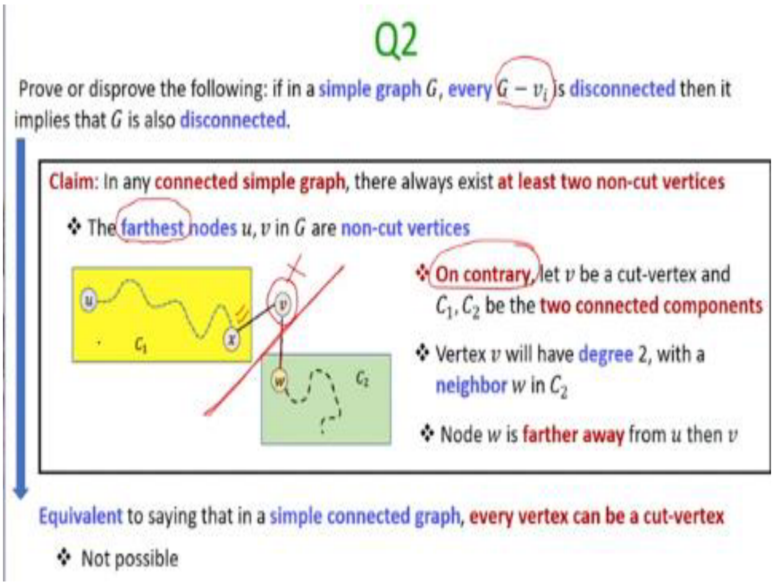

Now let us go to question number 2 here. We want to prove or disprove the following; If in a simple graph G, (G – v_i) is disconnected for every vertex v_i in the graph, then it implies that the graph G is also disconnected. Or equivalently here we want to check that if in the simple graph G, every vertex v_i is an articulation point or cut vertex, then the graph G is disconnected because if (G – v_i) is disconnected that means the vertex v_i is an articulation point.

So, we want to check whether this property is true or not. To prove this, let us prove a related statement or we prove a relative claim. So the claim is the following; If you take any connected simple graph then they always exist at least two vertices, none of which is an articulation point, this always holds in any connected simple graph. I can always guarantee the presence of two vertices which are not cut vertices, of course, the number of vertices in the graph is greater than = 2 because if the graph is just one vertex then this claim does not make any sense.

How exactly do we prove this claim? So, you focus on the nodes u, v in the graph G which are farthest, that means the distance among the nodes u and v is the maximum in the graph. That means you take all pair of nodes u, v find out a distance among those nodes u, v. And among all the (u, v) pairs, you focus on the (u, v) pair such that the distance is maximum in the graph G. My claim is that the nodes u as well as the node v are not articulation points, they are not cut vertices and this can be proved using a proof by contradiction.

So, on contrary, assume that say the vertex v is a cut vertex. So, this is without loss of generality the same argument can be applicable if we assume on contrary that the vertex u is a cut vertex. So, if the vertex v is a cut vertex, that means by deleting the vertex v, my graph gets divided into two connected components C_1 and C_2. And this means that the vertex v has degree 2 and it will have at least one neighbor say, w in the component C_2.

Because if the degree of the vertex v would have been just one that means if the only neighbor of the vertex v would have been this vertex x in the component C_1 and there is no neighbor w, then how can it be possible that deleting v disconnects the graph was split the graph into two component C_1 and C_2? So, since the deletion of v splits your graph into two connected components, that means there is something, some node w in C_2 such that v is having an edge to that node w in the connected component C_2.

But that gives you a contradiction to the fact that the nodes u and v are the farthest nodes in your graph G, because now you can see that a distance between the node u and w is more than the distance between the nodes u and v. That means by nodes u, v are not the farthest nodes in the graph, but it is rather the node u and w which are the farthest nodes in the graph. So, we get a contradiction and that proves that whatever we assume about the vertex v is not true, so we assume that a vertex v is a cut vertex, which is not true.

So, now coming back to the question, the question is equivalent to saying that can we have a simple connected graph where every vertex is a cut vertex? And that is not possible, because that is precisely what we proved in this claim. We proved in this claim that if at all your graph is a connected simple graph there are definitely and the number of nodes is greater than = 2, then they are definitely exist two vertices v_i and v_j such that neither v_i is an articulation point nor v_j is an articulation point, it is not possible.

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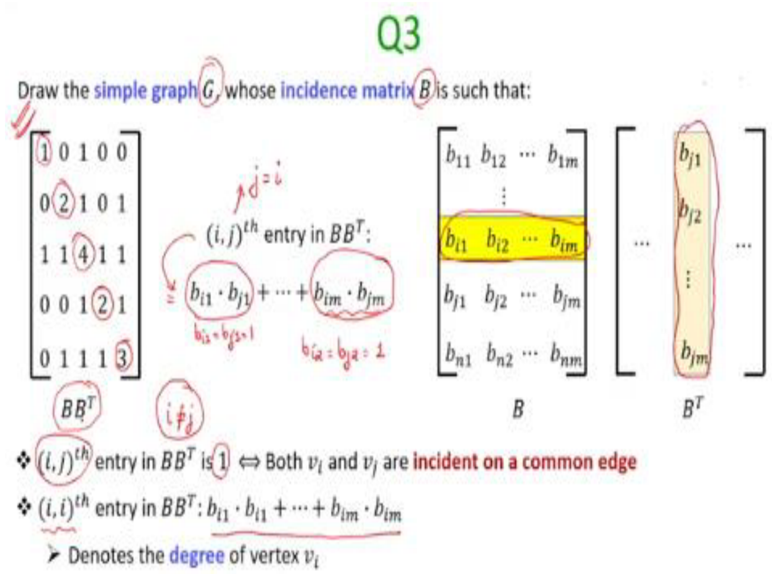

Let us go to question number 3. So here we have to find an unknown graph G, the graph G is not given to you but it is just given that it is a simple graph. And since the graph G is not known, we also do not know its incidence matrix B, but we know that incidence matrix B is such that the product of the incidence matrix and its transpose is this matrix. So, we have to basically recover the original graph G from the product of the incidence matrix and its transpose that is given to us.

Of course, a naive we have to intact will be you try all possible values of, for all cases of the B matrix and the B^T matrix and multiply them and check whether your guess gives you this value of BB^T matrix or not. We will not do that. We will argue and try to get back the information regarding the graph G. So, imagine that your number of vertices in the graph is n and the number of edges in the graph is m.

So, the incidence matrix of the unknown graph will be an n cross m matrix, so I am denoting the unknown incidence matrix by this notation. So, b_{11}, b_{12}, b_{1m} they are the unknown Boolean values, so remember each entry of the incidence matrix will be {0, 1}, either 0 or 1. And if the edge e is between the vertex v_i and vertex v_j, then in the incidence matrix if we focus on the eth row, then if we focus on the row number v_i and if you focus on the row number v_j and the column number e.

Then under the column e in the ith row we will have the entry 1 and the jth entry we will have entry 1 and all other entries will be 0. That is the property of your incidence matrix. Now, we do not know as of now which entries are 0, which entries are 1. Now it is easy to see that the jth row of the incidence matrix B will become the jth column in the transpose of the incidence matrix and that comes from the property of the transpose of a matrix. And what will be the (i, j)th entry, when we multiply the matrix B with the matrix B transpose?

So, how exactly the (i, j)th entry of the product of B and B transpose would be computed? We would have taken the ith row and we would have multiplied the ith row with the jth column component wise namely, b_{i1} would have been multiplied with b_{j1}. And then added to the product of b_{i2} and b_{j2}. And like that you would have multiplied the entry number b_{im}, the entry number b_{jm}, and if we had all these things that will give the (i, j)th entry of the matrix BB^T.

And we are given the value of BB^T. So now what I can say is the following; If I take any (i, j)th entry where i is not = j, then the (i, j)th entry in the product matrix BB^T will be 1 if and only if the vertex v_i and the vertex v_j are incident on a common edge. That means they are the endpoints of an edge. This is because we already argued that this is our (i, j)th entry, the (i, j)th entry is one and only if one of the values in this sum of m values is what because of none of the values.

So, if the first product in this sum is 0, and if the second product in the sum is also 0 and like that if the mth product in the sum is also 0 and how come the (i, j)th entry is one? So, (i, j)th entry is 1 only if you have $b_{i1} = b_{j1} = 1$, if that is the case that means the vertex i and vertex j they are the endpoints of the edge number 1 or b_{i2} should be = b_{j2} should be = 1, which implies that the ith vertex and jth vertex, they are the endpoints of the edge number 2, and so on.

So, if I focus on the (i, j)th entry, where i and j are distinct and checking whether they are 1 or 0, we can identify whether the vertex v_i and v_j are the endpoints of an edge 1. And you have that information available in the product matrix BB^T. And if I take the (i, i)th entry, that means if I substitute j = i here and focus on the (i, i)th then the (i, i)th the product matrix will be this expression and this is nothing but the degree of the vertex v_i.

So, you have all the information available now about the graph G in the product matrix. So your graph G is such that the degree of 1 is1, the degree of vertex 2 is 2, the degree of vertex 3 is 4, the degree of vertex number 4 is 2 and the degree of vertex number 5 is 3 and the endpoints of each edge is also available by focusing on the (i, j)th entry in this matrix, where i and j are distinct. So, this is how you can get back all the information about your unknown graph from the product matrix BB^T.

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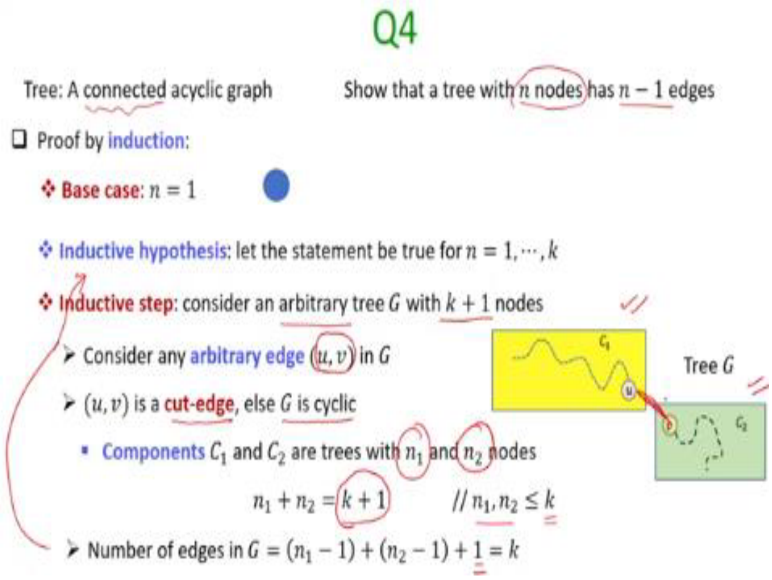

So, now let us go to question number 4. In question number 4 we have given the definition of a tree. So, a tree is a connected acyclic graph, that means it is a graph which is connected, that means you take every pair of distinct modes, there will be a path and it is acyclic, that means the graph has no cycle. We have to show that if you take any tree with n nodes, then the tree has n - 1 edges. So there are several ways to prove this theorem we will use proof by induction, induction on the number of nodes n.

So, the statement is obviously true for the base case namely for a tree which has only 1 node. So if you create a tree with 1 node then it will have 0 edges. Assume the inductive hypothesis is true, that means assume the statement is true for all trees consisting of up to k nodes. And now we are going to the inductive step, where we are going to consider an arbitrary tree consisting of k + 1 nodes. And we focus on any arbitrary edge with endpoints u and v in the graph G or the tree G.

My claim is that the edge (u, v) is a cut edge in the tree and this is true for any edge in the tree. You take any edge the claim means it will be a cut edge. That means if you remove the edge connecting the nodes u and v then your tree G gets splitted or divided into two connected components. If that is not the case, that means even if after deleting the edge between u and v, your tree remains connected that means there are still some way to get back to the node u from the vertex u even if this edge is not there between u and v.

Then we get the conclusion that there is a cycle in the graph but that goes against the definition of a tree. So, now if my edge (u, v) is a cut edge, I will get two components C_1 and C_2. I do not know how many nodes are there in C_1 and how many nodes are there in C_2? So, I can assume that they have n_1 and n_2 number of nodes respectively. But what I know is that if I sum up the number of nodes in C_1 and the number of nodes in C_2, that will give me the total number of nodes that I have in the tree, which is k + 1.

And I also know that both n_1 as well as n_2 are upper bounded by k. This is because C_1 and C_2 are disjoint and both of them are non empty, $\neq \phi$. So, since n_1 and n_2 is less than = k and both C_1 is connected as well as C_2 is connected, and I do not have any cycle in C_1 and I do not have a cycle in C_2. that means both C_1 as well as C_2 are individually trees with n_1 and n_2 nodes respectively.

So, as per the inductive hypothesis, I can apply the inductive hypothesis and claim that the number of edges in C_1 is n_1 - 1, the number of edges in C_2 is n_2 – 1. And that gives me the total number of edges in the original tree G is just one more than the number of edges that I have in the tree C_1 and C_2. This is because the only edge which I have touched or removed is the edge between the nodes u and v and that proves the inductive step.

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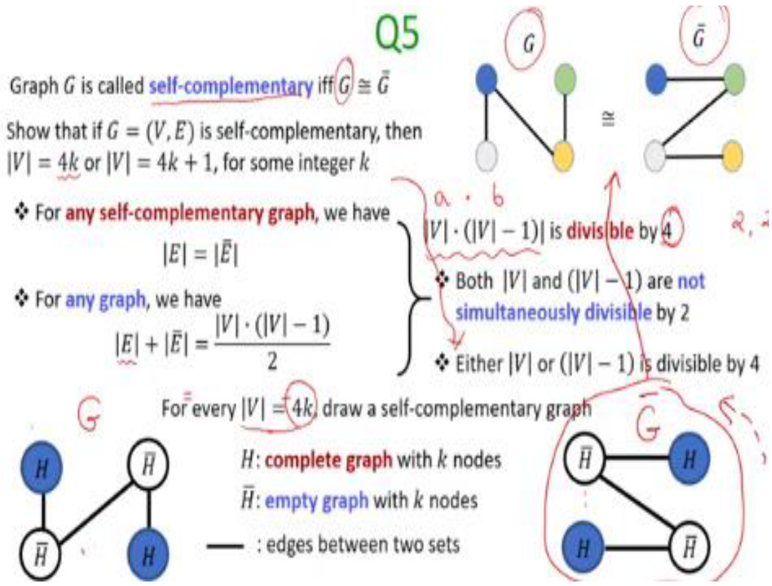

So, now let us go to question number 5. And question number 5, we define what we call as a self-complementary graph. So, a graph G is called self-complementary if it is isomorphic to its complement. So, for instance, this is a self-complementary graph with 4 nodes. I am not labeling the nodes but you can check here that the graph G and the graph $\bar{G}$, they are isomorphic to each other and hence the graph G is self-complementary here.

So, we want to prove here a very interesting property about self-complementary graphs, we want to prove that if your graph is a self-complementary graph, then the number of nodes in the graph is either a multiple of 4 or it is some 4 times k + 1. That means either the number of vertices is completely divisible by 4 or if you divide the number of vertices by 4 then you will get the remainder 1 that is what we want to prove here.

At least you can check that this statement is true for the G and G' that we have here; the number of vertices in G is 4 and 4 is visible by 4. So, how do we prove this? Since the graph G is self-complementary that means the number of edges in G and the number of edges in $\bar{G}$ have to be the same. So, the cardinality of the edge set E and edge set E complement will be the same and I know that for any graph it may not be self-complementary.

If you take any graph then the total number of edges in E and E complement is the same as the product of the number of vertices and the number of vertices minus 1 over 2. Because all the edges which are in G they would not be in $\bar{G}$ and vice versa and if you take the union of the graph G and the graph $\bar{G}$, you will get a complete graph with the number of nodes same as the number of vertices in the graph G or $\bar{G}$.

So, if you take $G \cup \bar{G}$ then you get the complete graph with n nodes, where n is the number of vertices in the graph G. So, remember the number of vertices in G and $\bar{G}$ will remain the same, you do not take complement with respect to the vertex set, the complement is respect to the edge set. So, we know these two facts, one fact that is true for every self-complementary graph and another fact which is true for every graph.

Based on these two things, if I substitute that cardinality of E is same as cardinality of E complement then I get that two times the cardinality of E is = $|E| + |\bar{E}| = \frac{|V|\cdot(|V|-1)}{2}$

which shows that the product of the cardinality of the vertex set and the vertex set minus 1 is a multiple of 4 or it is divisible by 4. Now you have 2 quantities here a and b and what are the prime factors of 4? 2 and 2, it turns out that you cannot have both a as well as b simultaneously divisible by 2.

Because one of the quantities is odd, then other quantity will be even. a and b you have you cannot have to consecutive numbers both of them simultaneously to divisible by 2 and but since I know that the product of a into b is divisible by 4, then that is possible only if either a is divisible by 4 or b is divisible by 4. Then both a and b would have been individually divisible by 2, then I cannot claim this fact here.

But since I know that both a and b cannot be simultaneously divisible by 2, but the overall product a times b is divisible by 4 then that is possible only if a times b is divisible by 4. It only then either a is divisible by 4 or b is divisible by 4 and that shows what we wanted to prove here. Now in the same question we want to draw a self-complementary graph with a vertex set which has 4k number of nodes.

So, you are given a value k, k could be anything, it could be 1, 2, 3, 4 given the value of k you have to draw a self-complementary graph, which has 4 times k number of nodes. So, if k is = 1, then this is the self-complementary graph, but you cannot draw a distinct self-complementary graph for each and every value of k, I just want to draw a general graph which unifies all self-complementary graphs with 4k number of nodes, so how do we do that?

So, what I do here is I take four groups of k nodes and those four groups of k nodes are disjoint. So this is my first group, this is my second group, this is my third group and this is my fourth group. Now this group of k nodes denotes a complete graph with k nodes, that means you have an edge between every pair of distinct nodes in this group and similarly this copy of k nodes denotes a complete graph with k nodes.

And this group $\overline{H}$ denotes an incomplete graph of k nodes, that means it is a collection of k nodes with zero edges and similarly this collection of k nodes have zero edges. Now what I do is the following my G is the following my graph G which is a self-complementary graph with four k nodes is the following, so of course my graph G will be now having 4k nodes because the total number of nodes are k + k + k + k, so 4k nodes.

Now what I am doing here is the following; So, this thick edge between this group of k nodes and this group of k nodes denotes that if you have the nodes v_1 to v_k here and if you have the nodes v_{k+1} to v_{2k} here this collection, then you have an edge between every node in this group and every node in this group. That is what is the interpretation of this thick edge. Similarly you have an edge between every group between every node in this group and every node in this group.

That is interpretation of this thick edge and similarly you have an edge between every vertex in this group and every vertex in this group, that is interpreted by this thick edge, that is interpretation of this thick edge, so that is my graph G. Now what will be the complement of this graph G? So, this is my graph G, so my graph $\bar{G}$ will also have 4k number of nodes but then what will happen is the following.

So, since this group was a group of k nodes with edges between every pair of nodes you will still have those k nodes but no edges among any pair of nodes, whereas this copy of k nodes will get converted into a complete graph of k nodes. This copy of a complete graph of k nodes will get converted into a collection of k nodes with no edges and this copy of a graph with k nodes and zero edges gets converted into a complete graph with k nodes. And then since these edges are present in G, they will not be present here anymore.

So, that is why these edges have vanished and similarly you can see the edges which were not there in G there will be now present in $\bar{G}$ and vice versa. The edges which were not there in G they will be present in $\bar{G}$, so there were no edges between this group and this group in G, but now those edges are here and so on. So, that is how your $\bar{G}$ will look like, so it is easy to see that your graph $\bar{G}$ is isomorphic to the graph G.

I can interpret or rearrange or redraw the graph $\bar{G}$ in the same form as the graph G. I just have to orient it a little bit, that is all, say if I orient this graph rotate, this graph little bit like this, then I get the same structure as the graph G and that shows that my graph G and G is self-complementary because it is isomorphic to its own complement.

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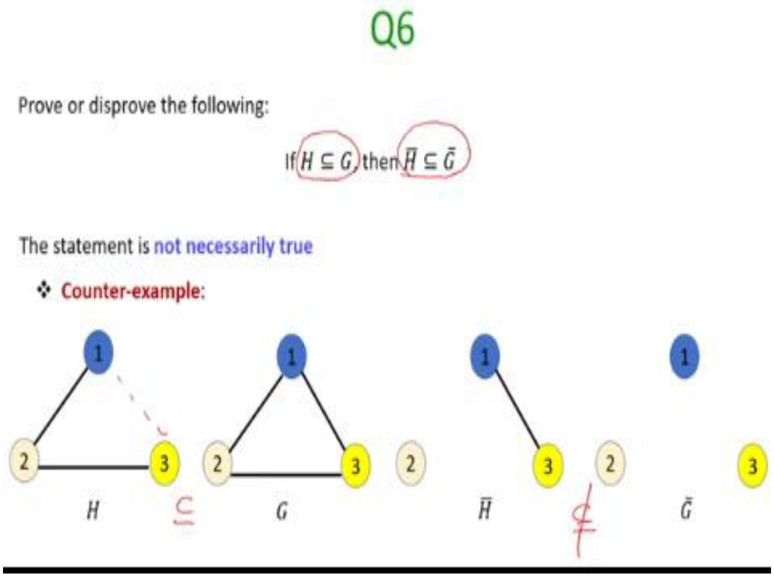

In question 6, we have to either prove or disprove the following; If the graph H is a subgraph of G then can we say that $\bar{H}$ is also as the graph of $\bar{G}$. Well, the statement is not necessarily true. A very simple counter-example is the following: here you have a graph H and a graph G, the graph H is a subgraph of G, but if you take $\bar{H}$, in $\bar{H}$, the only the edge will be between the nodes 1 and 3 because the edge is between 1 and 3 was not there. Where in $\bar{G}$ there will not be any edge, so clearly H prime or the graph $\bar{H}$ is not a subgraph of the graph $\bar{G}$. So, this statement is not necessarily true.

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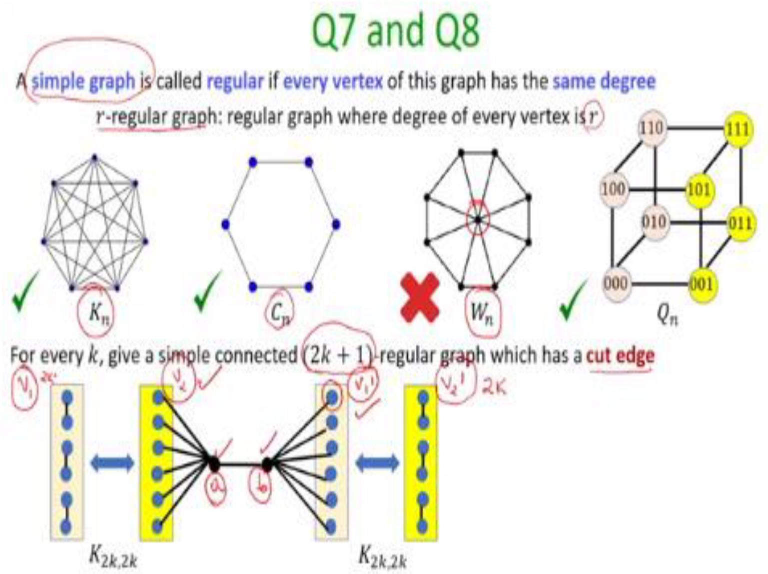

Then let us see question number 7 and 8: a simple graph is called regular if the degree of every vertex is the same. And if the degree of every vertex in a simple graph is some value r, then we call such a graph an r-regular graph. So, here you are given a few graphs and we have to find out which of these graphs are regular and which are not. so K_n is a regular graph because the complete graph with n nodes in such a graph the degree of every vertex is n - 1, so it is regular.

The cycle graph with n nodes is also regular because if you take the degree of every vertex, it will be 2. But your wheel graph W_n is not a regular graph, because it is the central node which has a huge degree compared to the other vertices of the graph. Whereas if you take the hypercube graph, we can prove that it is a regular graph and the degree of every vertex will be the same.

Now what we have to do in question 8 is the following; You are given a value of k. You have to draw a simple regular graph where the degree of every vertex is 2 times k + 1 such that the graph has a cut edge. You have to give the construction of a general graph. So, what we can do here is the following; we take two copies of a complete bipartite graph. So we take one copy of a complete bipartite graph where I have 2k nodes in the individual sets in the bi-partition.

So, 2k number of nodes in v_1 and 2k number of nodes in v_2 and I have an edge between every vertex in v_1 and every vertex and v_2, that is denoted by this arrow symbol, bi-implications symbol. Because the graph will look very ugly if I keep on adding edges between every vertex in v_1 and every vertex and v_2. So, to avoid making it look ugly I just denote the existence of an edge between every node in v_1 and every node in v_2 by this bi-implication.

And similarly, I take another copy of a complete bipartite graph and say I call the bipartition of this copy of the complete bipartite graph as C_1' and the C_2' and again I have an edge, I have an edge between every node in v_1' and every node in v_2'. So, this is my v_1, v_2, v_1', v_2'. Now I need to ensure that my graph has a cut edge. So I try to introduce a cut edge. This will be my overall cut edge and what I do here is, so let me call the end points of this cut edge as a and b. I connect the node a to every vertex in the subset v_2. So, that will ensure that the degree of the vertex a is 2k + 1, why 2k + 1? Because it will be connected to all the vertices of v_2, so it gets degree 2 k through that and it is also having an edge to the node v so that ensures that a degree of a is 2k + 1.

Using the similar argument, I can say that a degree of b is also 2k + 1 because b has a neighbor in every vertex with v in v_1', so through that it gets 2k degree and b is also a neighbor of the node a, so an additional degree, so the degree of b is also 2k + 1. Now all the vertices of v_2 will have degree 2k + 1. This is because you take any vertex of v_2, say the first vertex, so it is a neighbor of a degree 1 and it is having an edge with every node in v_1.

So, through that it gets the degree 2k, so total degree 2k + 1. Due to the same argument each node in v_1' also have degree 2k + 1 because you take any node in v_1', it is having an edge with every node in v_2', so that it gets degree 2k and the same node is also having an edge with the node b, so through that it gets one more degree so total degree 2k + 1. But we also need to ensure that every vertex in v_1 and every vertex in v_2' also gets degree 2k + 1.

Till now we have ensured that the degree of a is 2k + 1, we have ensured that the degree of b is 2k + 1, we have ensured that every vertex in v_2 has degree 2k + 1 and we have ensured that every vertex in v_1' has degree 2k + 1. But right now the degree of every vertex in v_1 is 2k and similarly the degree of every vertex in v_2' is 2k. I need to increase the degree of each vertex in v_1 by 1 and each I have to increase the degree of each vertex in v_2' by 1 as well and that is simple.

What we can do is the following; Take the vertex set v_1, you pair them into k pairs, you take the first two nodes and add an edge between them, then you take the third and the fourth note and add an edge between them and like that you take the fifth and sixth node and add an edge between them and so on. So, that will ensure that the degree of every vertex in v_1 becomes 2k + 1, and if you do the similar process for v_2' as well. That will ensure that the degree of every vertex in v_2' becomes 2k + 1. So, with that I conclude the tutorial number 8. Thank you!