Discrete Mathematics
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Lecture -39
Solving Linear Non- Homogeneous Recurrence Equations

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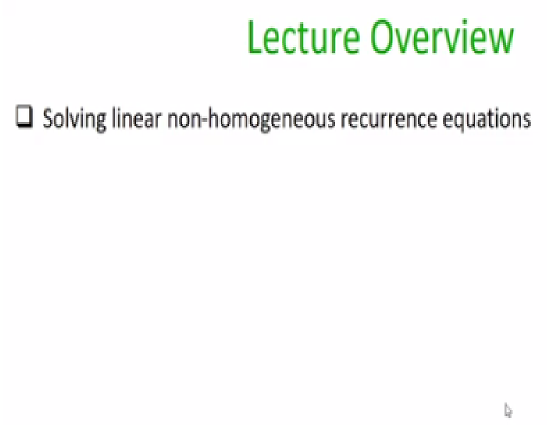

Hello everyone welcome to this lecture. So, in this lecture we will continue our discussion regarding how to solve recurrence equations, linear recurrence equations. And in this lecture we will focus on how to solve linear non-homogeneous recurrence equations.

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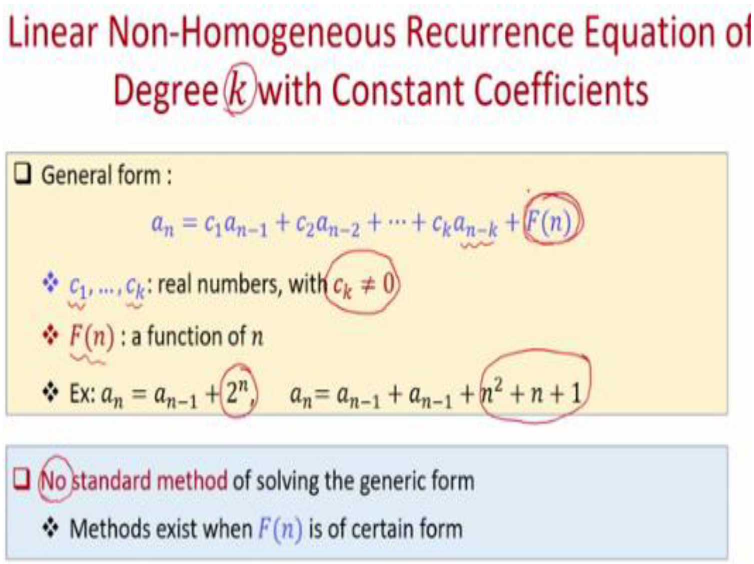

So, let us first discuss the general form of any linear non-homogeneous recurrence equation of degree k with constant coefficients. So the general form will be this, the nth term will depend on previous terms plus some function of n, F(n). So, here your coefficients c_1 to c_k are real numbers they could be 0 as well but the only restriction is that c_k is not allowed to be 0 that means the nth term definitely depends upon the $(n - k)^{th}$ term.

And that is why the degree of this equation is k. And F(n) will be a function of n that is why it is called non-homogeneous recurrence equation. So, some examples of recurrence equation in this category are as follows. So, in this equation your F(n) is 2^n and this equation your F(n) is $n^2 + n + 1$ and so on, it turns out that unlike linear homogeneous recurrence equations of degree k for which we have a standard method of finding the general form of any solution, we do not have any standard method for solving the non-homogeneous recurrence equation because we do not know what could be the structure of this function F(n). But it turns out that for some specific form of this function F(n) we have some well-known methods and in this lecture we will discuss those methods.

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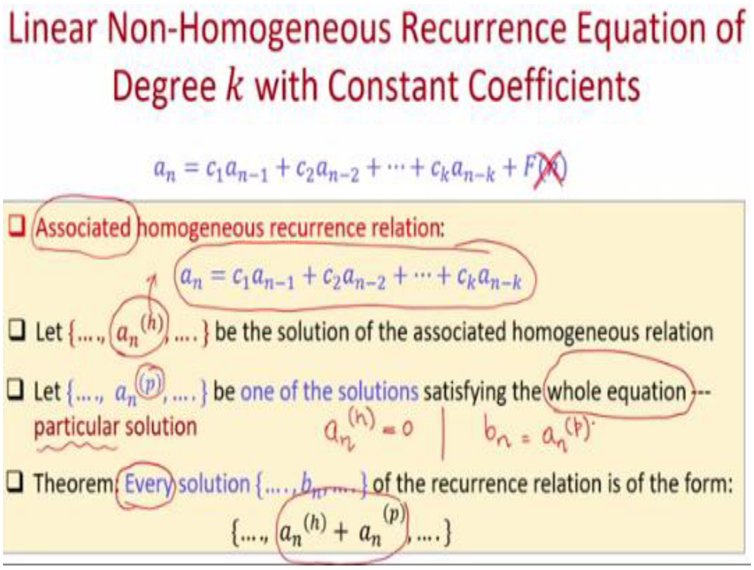

So, the first thing that we do, while solving the linear non-homogeneous recurrence equation is the following. We form what we call as the associated recurrence relation, associated homogeneous recurrence relation to be more specific and this is obtained by chopping off this F(n) function. So, if I chop off this F(n) function then whatever recurrence relation I am left over with that is called as the associated homogeneous recurrence relation.

It will be of degree k and then we know how to solve this. We have seen extensively in the earlier lecture how to solve a linear homogeneous recurrence equation of degree k, the general form of it can be obtained by using those methods. So, let the solution be denoted by the sequence whose nth term is $a_n^{(h)}$. So, this h denotes that the sequence is the solution of the associated homogeneous sequence relation. It may not satisfy the entire recurrence equation.

So, remember we have to solve or we have to find out a sequence satisfying the entire recurrence equation where F(n) is also a part of the equation, but the sequence $a_n^{(h)}$ is a solution only for the associated homogeneous recurrence relation. And then what we do is the following, we try to find out one of the solutions satisfying the whole recurrence equation. Namely a sequence satisfying the entire recurrence equation and we call that solution as a particular solution.

So, the sequence satisfying the entire recurrence equation which is the particular solution the nth term of it is denoted by $a_n^{(p)}$. Then the claim is that any sequence which satisfies the entire recurrence equation; its nth term will be the summation of the nth term of the sequence satisfying the associated homogeneous equation and the nth term of the particular solution; that is the statement.

So, what basically we are trying to say here is that once you have obtained one of the solutions satisfying the entire recurrence equation,then you can express any solution satisfying the entire recurrence equation in terms of that particular solution or one of the solutions that you have obtained and the solution for the associated homogeneous reference equation. By the way now if you substitute $a_n^{(h)}$ to be 0 then you get automatically that $b_n = a_n^{(p)}$ is also one of the solutions satisfying the entire recurrence equation.

So, we can derive any solution satisfying the entire recurrence equation from this generic solution.

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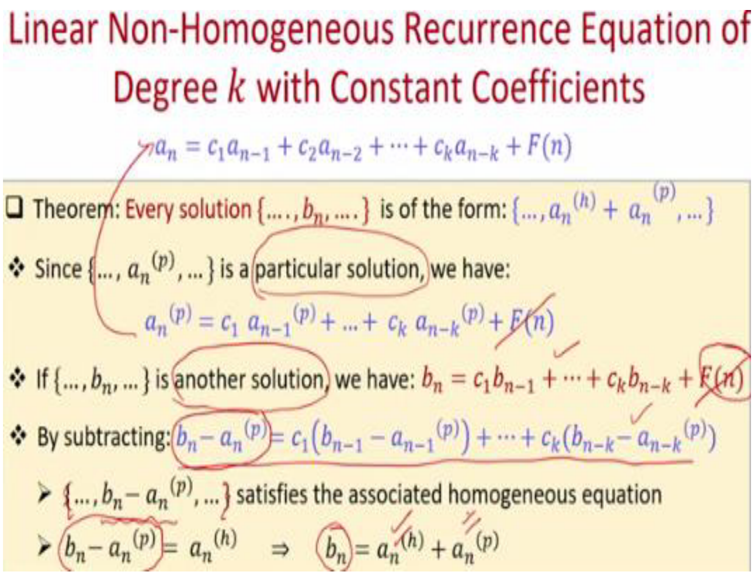

So, let us prove this theorem, we want to prove that any solution satisfying the entire recurrence equation is of this form. Namely its nth term is the summation of the nth term of the homogeneous recurrence equation or associated homogeneous sequence equation plus nth term of the particular solution that is what we want to prove. So, since $a_n^{(p)}$ is one of the solutions satisfying the entire recurrence equation, we can say that its nth term is c_1 times the n -1 term of that sequence + c_2 times the n - 2 term of that sequence.

And like that c_k times the (n – k) term of that sequence + F(n), because that is what is the implication of saying that this particular solution satisfies the entire recurrence equation. And now if there is another sequence whose nth term is b_n and that is also one of the sequences satisfying the entire recurrence equation, then we get the implication that b_n is equal to c_1 times $b_{(n-1)}$ + c_2 times $b_{(n-2)}$ + … + c_k times $b_{(n-k)}$ + F(n).

Then what we can say is the following, if I subtract the nth term of the particular solution and the nth term of the b sequence then the effect of F(n) and F(n) cancels out and we get this property. And now how can you interpret this property? You can interpret this property as if you have a sequence whose nth term is the difference of the nth type of the b sequence and the particular solution and this sequence is now satisfying the associated homogeneous recurrence relation.

Why so? Because if you take this property here, this property basically says that this b_n - $a_n^{(p)}$ is equal to c_1 times ($b_{(n-1)} – a_{(n-1)}^{(p)}$) + up to that like that c_k times ($b_{(n-k)}$ - $a_{(n-k)}^{(p)}$). So, you can imagine that you have now a sequence satisfying the associated homogeneous recurrence equation because the associated homogeneous equation is that a_n the nth term should be equal to c_1 times (n – 1) th term + c_2 times (n - 2) th term + … + c_k times (n – k) th term.

And indeed you have a sequence satisfying this recurrence relationship and the nth term of that sequence is basically b_n - nth term of the particular solution. So, based on this what we can say is the following, we can say that the nth term of the sequence which is basically the difference of the nth term of the b sequence and the nth term of the particular solution is a solution for the associated homogeneous equation.

And since we have used the notation $a_n^{(h)}$ for denoting the solution for the nth term of the sequence satisfying the associated homogeneous equation, we get the implication that any sequence satisfying the entire recurrence relation; its nth term is basically the summation of the nth term of the sequence satisfying the associated homogeneous recurrence relation plus the the nth term of the particular solution satisfying the entire equation, so that proves our theorem.

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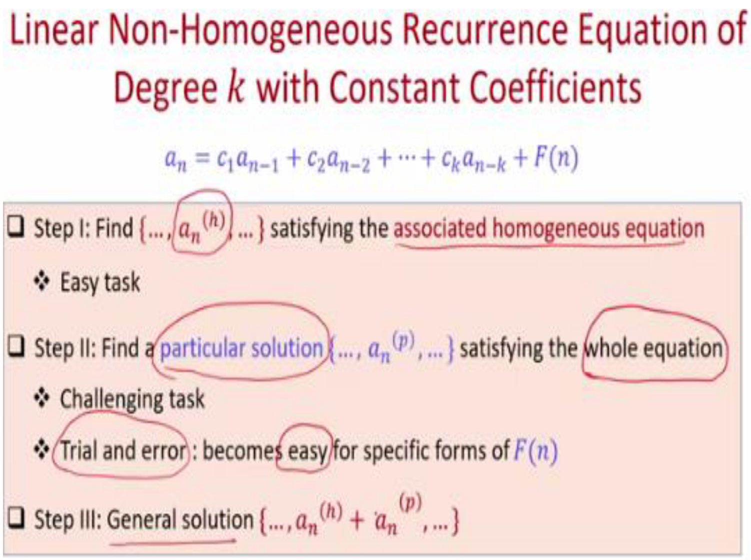

So, coming back to the method for how to solve linear non-homogeneous recurrence relation the first step will be to solve the associated homogeneous equation which is easy. Because the associated homogeneous equation will be of degree k and we will know how to solve it because we have seen extensively how to solve linear homogeneous recurrence relation of degree k. So, say the general form of the solution satisfying the associated homogeneous equation is this.

The step two is the difficult part, namely coming up with a particular solution satisfying the whole equation, and this is the challenging part of, coming up with the general solution satisfying the entire equation. So, remember we want to find out the general form of any solution that satisfy the entire recurrence equation and for that we need one of the solutions satisfying the entire equation.

You might be wondering that if we can find out one of the solutions satisfying the entire equation why we are bothered to find other solutions satisfying the other equation. That this is because we want to find the general formula which covers all possible solutions all the sequences satisfying the given recurrence condition. Remember a recurrence equation can have infinite number of solutions if I do not give you the initial conditions.

So, I would like to find out a general formula, a general form of the solution which covers all possible sequences satisfying the given non-homogeneous recurrence equation. Just finding one of the solutions is not sufficient, that is one of the sequences satisfying the entire equation but I might be interested to find out other sequences as well and that is why we need the particular solution.

So, for finding this particular solution we do not have any well known methods or rules. What we do is to try to find out the particular solution by using what we call as trial and error and this trial and error method becomes easy for some specific forms of this function F(n). We will see those specific forms and assuming that we have obtained a particular solution then we can write down the general solution as a summation of the associated homogeneous equation and the particular solution. That is a method for solving linear non-homogeneous recurrence equations.

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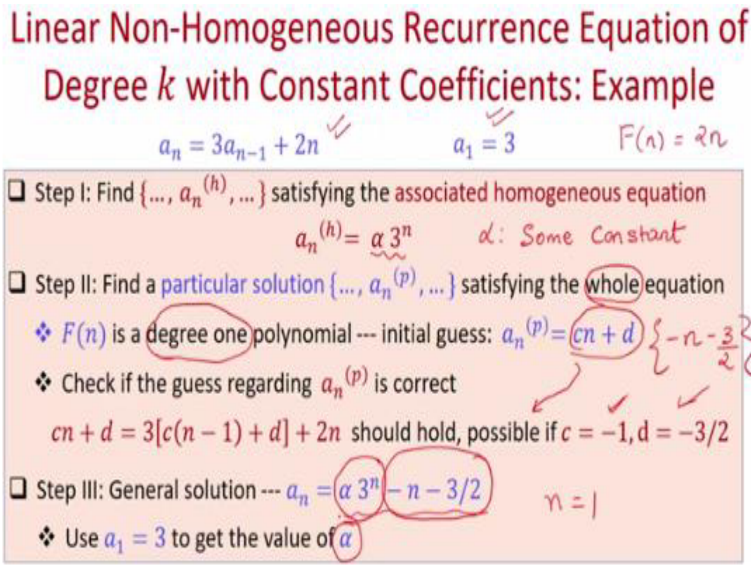

So, now let us see how we can find out the particular solution for some specific forms of F(n) function using the trial and error method. So, let us take this example where my F(n) is 2n and the associated homogeneous equation is of degree 1, So, I can solve it, the characteristic equation will have only one root namely 3. So, the general form of the associated homogeneous equation will be this where α is some constant, unknown constant.

And now we have to find out the particular solution satisfying the whole equation. So, for that I will make some guess about the particular solution. So, in this case I observe that my F(n) is a polynomial of degree 1, because my F(n) here is 2^n which is a polynomial of degree 1. So, I make a guess that let my particular solution be a polynomial of degree 1 for some constant c and d. I do not know the exact values of c and d I am just guessing that let this be the particular solution : cn + d.

And now I have to check whether my guess is correct or not about the particular solution. How do I check whether my guess regarding the particular solution is correct or not? I have to check whether there exist values of c and d, such that if I substitute those values of c and d in my guessed particular solution then it satisfies the entire recurrence equation. So, let us do that, In order that cn + d or a sequence whose nth term is cn + d satisfies the entire recurrence condition this relationship should hold.

Namely the nth term is this, should be equal to 3 times the (n – 1) term and the (n – 1) term as per my guess of the particular solution will be this plus the function of n, and this relationship will hold if c is equal to - 1 and d is equal to – 3/2. How do I get these values of c and d? Well basically I rearrange the terms here, and then compare the LHS part and RHS part and based on the comparison I come up with these values.

That means it turns out that if indeed c is equal to - 1 and d is equal to – 3/2, then I have a particular solution namely a sequence whose nth term is (– n – 3/2) and this sequence satisfies the entire recurrence equation. So, I am successful in finding the particular solution; successful in the sense my guess about the particular solution here is correct. And then I will say that my general solution satisfying the entire recurrence equation namely any solution satisfying the entire recurrence equation will be of this form: it will be the summation of the nth term of the sequence satisfying the associated homogeneous equation which is some constant times 3^n, plus the nth term of the particular solution which I am able to find out using the trial and error method. So, this will be the general form of any solution. Now if you are interested to find out the unique solution satisfying the entire recurrence equation as well as the initial condition.

You have to substitute n equal to 1 here and by substituting n equal to 1 here you will be able to find out the value of the exact constant α satisfying the initial condition as well as the entire recurrence condition.

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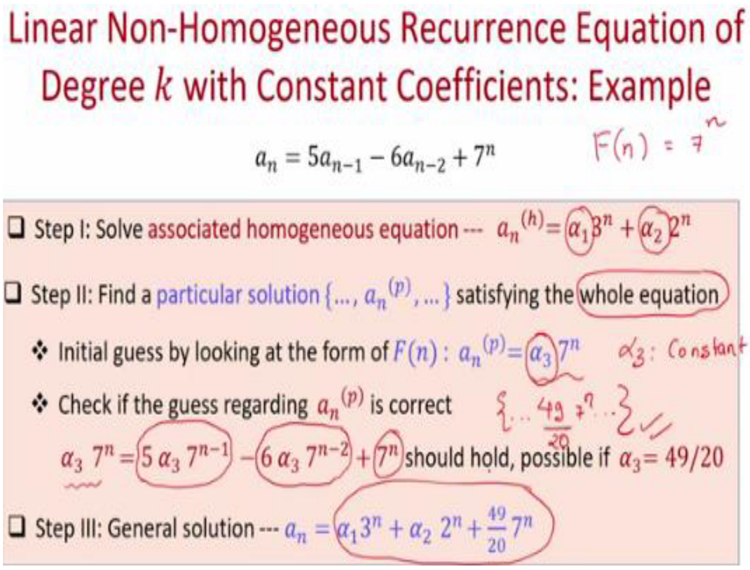

Now let us see another case or another structure of F(n) function. So, in this case my F(n) function is some constant power n. So the step one will be solving the associated homogeneous equation, so it will be an equation of degree 2. So, the characteristic equation will have two roots and the two roots are distinct. So the general form of the solution for the associated homogeneous equation will be some constant times 3^n plus another constant times 2^n.

Then my goal will be to find out the particular solution satisfying the entire recurrence equation. So, what I do is in this case as I said my F(n) is some constant and the constant is 7^n, I make a guess that let my particular solution be some constant times 7^n. So, α_3 is now a constant here. So, I am using different constants I am using α_3 here as a different constant to distinguish it from the constants α_1 and α_2 which are there as part of the solution of the associated homogeneous equation.

Now I have to check whether indeed my guess about the particular solution is correct or not, how do I check that? Well I have to see whether I can find out the value of α_3 such that, that value of α_3 times 7^n is a solution for the entire recurrence equation. So, I have to check whether there exists a value of α_3 such that a sequence whose nth term is α_3 times 7^n satisfies the condition that it is equal to 5 times the (n – 1)th term.

So, the 5 times (n – 1)th term for that particular solution or the guessed particular solution will be this minus 6 times the (n – 2)th term of the guessed particular solution plus 7^n. And then again if I rearrange the terms I see that indeed it is possible to have a value of α_3 namely if α_3 is 49/20 then I can say that a sequence whose nth term is 49/20 times 7^n satisfies the entire recurrence condition.

That means in this case again I am able to successfully find out a particular solution and now the rest of the steps are simple. I will now say that the general solution of any sequence satisfying the entire recurrence condition will be this; the nth term will be this. So, here the unknowns will be now α_1 and α_2 if you are given two initial conditions for the given recurrence equation then by substituting n = 1 and n = 2 you can find out the exact values of α_1 and α_2.

But if you are not given the initial conditions then you will say that any sequence satisfying the entire recurrence condition will be of this form.

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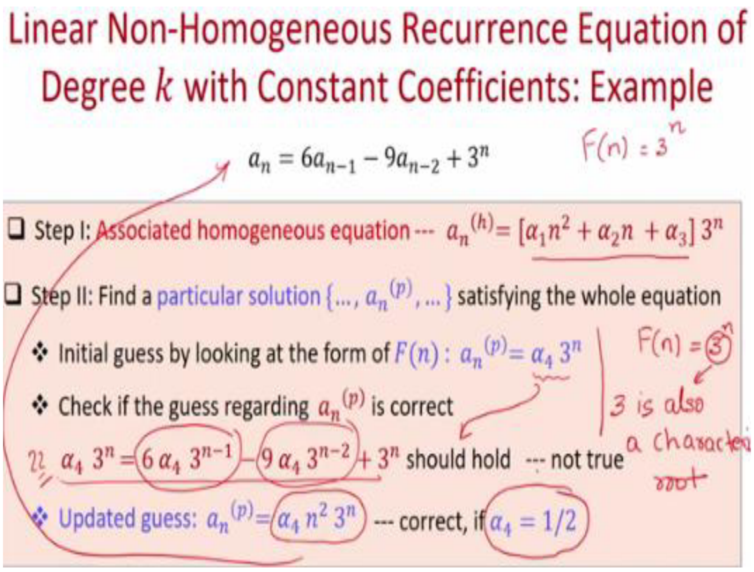

Now let us see another example. Here my F(n) is 3^n, so I will first solve the associated homogeneous equation. The solution will be this, this is because 3 will be the characteristic root and it will be repeated two times. So that is why the general form the solution for associated homogeneous equation will be an unknown polynomial of degree 2 followed by the characteristic root raised to power n.

Step 2: I have to find out a particular solution. So as I did in the previous example my guess will be that the particular solution is some constant times 3^n. And now if I proceed to check whether my guess about a particular solution is correct or not, I have to check whether I can find out the value of α_4 such that this condition holds, namely the nth term of this particular solution should be equal to 6 times the (n – 1)th term of the particular solution - 9 times the (n – 2)th term of the particular solution + 3^n.

And now if I rearrange the terms and try to solve and come up with the value of α_4 you will see that I cannot find the value of α_4. There exists no value of α_4 such that this condition holds, I cannot do that. Then where I am going wrong, it worked for the previous example where my F(n) was 7^n but then why it is not working here? Well the reason it is not working here is that your F(n) is 3^n and 3 is also a characteristic root.

Whereas in the previous example my F(n) was 7^n but 7 was not a characteristic root, it turns out that if I now make a guess that my particular solution is some constant times (n^2 3^n). And then try to check whether this satisfies the recurrence condition or not, namely whether I can find out the value of α_4 such that it satisfies the given recurrence condition I will be able to find out the value of the constant α_4. Namely by rearranging the terms you will see that α_4 being 1/2 is a valid solution.

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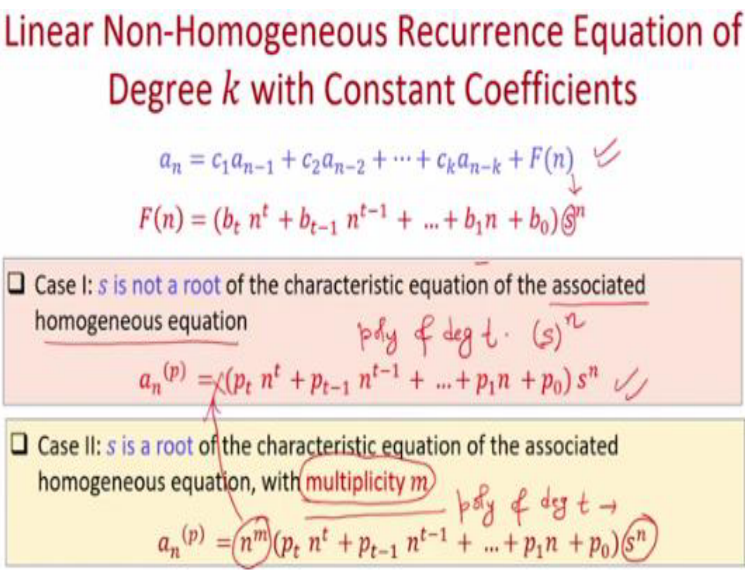

So, now let us unify all the examples that we have discussed till now and come up with the general theorem statement. So, the claim is the following, imagine you are given a linear non homogeneous equation of degree k and suppose your F(n) function is of the following form, it is a polynomial of degree t and some constant power n. Suppose it is of this form then we have to check the following.

We will check if this constant s is a root of the characteristic equation of the associated homogeneous equation or not. So, remember the step 1 for solving the non-homogeneous recurrence equation will be to solve the associated homogeneous equation. And when we try to solve the associated linear homogeneous equation we will be forming a characteristic equation so it will have characteristic roots.

So, we have to check whether the constant s which is occurring in the function F(n) is one of those characteristic roots or not. So, it could be either a characteristic root or it may not be a characteristic root. So, if it is not a characteristic root then the theorem states here that the particular solution of the form: a polynomial of degree t followed by the same constant s^n is a valid particular solution.

Namely any sequence whose nth term is this value will satisfy the entire recurrence equation; this is for the case where the constant s is not a characteristic. But if it is a characteristic root then depending upon how many times that root is repeated in the characteristic equation; that suppose if s is a root and that too with multiplicity m where m is greater than equal to 1 then the general form of the particular solution will be the following.

You still have a polynomial of degree t and then you also have s^n but you also now need n^m. So, that is the difference between case 1 and case 2. In case 1 you do not have this n^m this is not there, because in case 1, s was not occurring as a characteristic root but in case 2, s is occurring as a characteristic root and that too m number of times if m is equal to 1 then you will have n^1; if m is 2 that means s is occurring as a root 2 times then it will be n^2 and so on.

And we can prove this easily, I am not going into the exact proof you can check that easily; these are the two cases.

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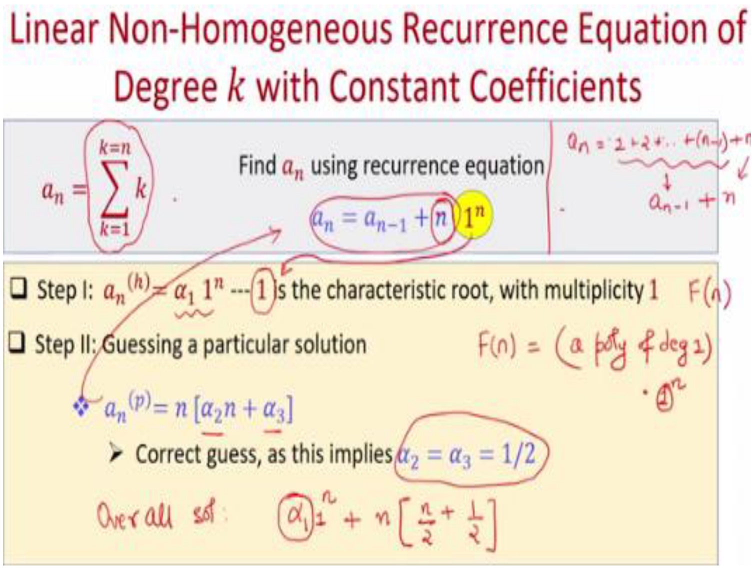

And now let us see; let me demonstrate the application of this general form with some specific examples. So suppose I want to find out the nth term of this summation. So, this a_n here basically denotes the sum of first n natural numbers; basically a_n here denotes the sum of the numbers 1 to n so easily it is easy to see that a_n is a sequence depending upon what is the value of n you get different values.

So, we have to find out the formula for the nth term of the sequence and that too using recurrence equation. So let us first formulate the recurrence equation. It is easy to see that a_n is nothing but the summation of the (n – 1)th term of the sequence plus n, because your, a_n is 1 + 2 + … + n - 1 + n. Now the summation 1 to n - 1 is nothing but $a_{(n-1)}$ and the + n is carried over. So, now this is a linear non-homogeneous recurrence equation of degree 1.

So, let us solve it. So we will first solve the characteristic equation which will be of degree one it will have only one root and in this case the root is 1 and its multiplicity is 1. So, that is why the general form of the solution for the associated homogeneous equation will be some constant times 1^n and now we have to come up with a particular solution. So, what is the general, what is the function F(n) here?

You might be saying that F(n) is n but that is not the case, you have to be very careful here even though explicitly you will see that F(n) is just n, but I can always say that there is an implicit 1^n which is there in the F(n) function and why I am taking this implicit 1^n ? this is because I am in the case where this constant 1 which is implicitly present in F(n) is also occurring as one of the characteristic roots.

Otherwise I would not have considered this implicit 1^n. I could have simply ignored it, but since one is a characteristic root and the same constant 1^n is occurring in the function F(n) I have to be careful. So, now if I use the general form; if I use the result that I stated for the general form of the particular solution my F(n) here in this case is a polynomial of degree 1 multiplied by 1^n.

And this 1 is a characteristic root with multiplicity 1 so that is why there will be n^1 outside in the particular solution followed by a polynomial of degree 1, followed by 1^n. Now this 1^n I can ignore. So, this will be the general form of the particular solution and now I have to find out the values of α_2 and α_3 satisfying the particular solution that means I have to check whether my guess regarding the particular solution is correct or not.

For that I have to check whether this guessed particular solution satisfies the entire recurrence equation; for that I have to check whether $a_n^{(p)}$ is equal to $a_n^{(p-1)}$ + (n * 1^n), check this and try to find out the values of α_2 and α_3 and it turns out that by solving and rearranging the terms I will get the values of α_2 and α_3 that means my guessed particular solution is correct.

And now if I have the guessed particular solution I can say that the overall solution will be the summation of the nth term of the associated homogeneous equation plus the solution for the nth term of the particular solution. So, this will be the overall solution. If you want to find out the exact value of this constant α_1 you can use the fact that a_1 is 1; you can have an initial term and then substituting n equal to 1 you can find out the exact value of this constant α_1.

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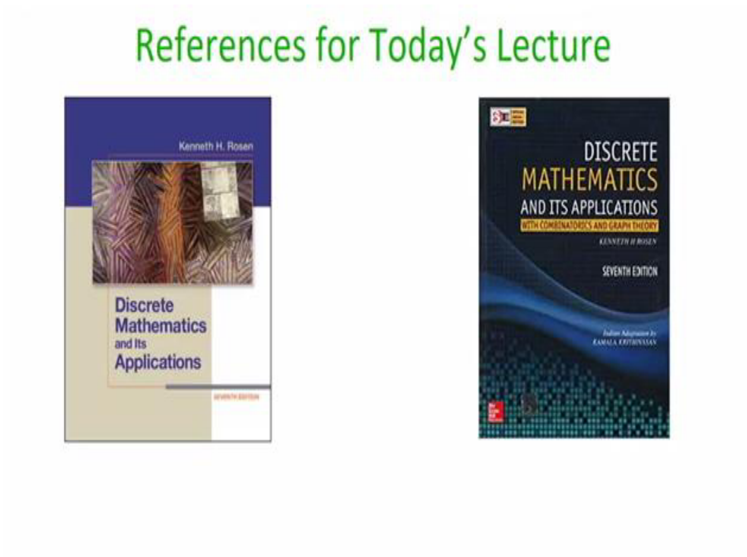

So, that brings me to the end of this lecture so these are the references for today's lecture just to summarize in this lecture we discussed how to solve linear non-homogeneous recurrence equations of degree k. The general solution is obtained by solving the associated linear homogeneous equation and getting a particular solution. And coming up with the particular solution is done by a trial and error method but it becomes methodical if you have the F(n) function in some specific form, thank you.