Fluid Mechanics
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Lec 33: Boundary Layer Approximation III

Good morning. Let us discuss about boundary layer approximations what we have been discussing on in the last two classes. In the last class we have derived boundary layer equations which is the approximations of Navier-Stokes near to a boundary layer formations to a flow past in a flat plate which is a very simplified conditions. That is what we have discussed. Today we will look at more detailed way how we can get the solutions of a boundary layer equations, numerical solutions of boundary layer equations. So mostly I have been following it the book of Senzel Simbala book.

So more detailed derivations you can look it Senzel Simbala book and as I said it earlier we are talking about the introductions of boundary layer approximations because we are not going details in these undergraduate course levels. But if you are really interested about boundary layers there are very good books are available where you can look at the concept of boundary layers okay. That is what is I think to professors, okay, Prandtl and his PhD students, okay, Paul Richards and Blasey, they worked hard early 90s to look for solutions for the boundary layer approach, that is their contributions. Today, we will talk about the laminar boundary layers, laminar boundary layers solutions.

Then we will talk about a concept of displacement thickness, momentum thickness and introductions at the flat plate boundary conditions for the turbulent flat plate boundary layers. So this is what the contributions from Brendel as well as the places okay. So those they have contribute early 1900s where that time there was no computers what we have today. So how they try to solve this the boundary layer problems to estimates the velocity distributions and the boundary layer thickness. Looking that let us go for the BC concept what we are talking about.

we are talking about laminar boundary layers equations okay. So as you remember it the boundary layer equations what we derive it first is the mass conservation which is very simple as again I am repeating it for two-dimensional incompressible flow velocity divergence which is equal to del u by del x del v by del y. That is what you just simply remember this part which is very easy to understand. Remember it that the boundary layer mass conservation equations. So if I look it as we derive it the linear momentum equations linear momentum equations and Bernoulli's equations both we combine it.

to get it the boundary layer equations in this form the basically x direction equations for steady incompressible flow. That is what is coming out to be as you know it del u by del ux v del u by del y is equal to we have u dy dx, this is the free stream velocity plus you have mu del square y square okay. So that is the equations of boundary layer equations. So this is the equations which is a parabolic form okay, parabolic equations. as compared to the Navier stoke equations solving these two equations are easy because you know it from these two equations we can get a solutions of u and v and from that solutions we can get it what will be the layer thickness, what will be the wall shear stress, what will be the screen fixants or drag coefficients that is what our idea and what is the thickness of the boundary layers.

So if you look at that this is a parabolic equations which simplified us just trying to sketch that part okay. So that means if I have a wall surface like this and you have a formations of boundary layers, that is what is sigma x, okay. So that means if I take a control volumes, I know these boundary conditions at the wall is the u v both are the 0 that is the non-slip boundary conditions. And at this point at the inlet point okay at the section 1 1 I know this the velocity distributions okay that is what we know the velocity distributions. Also we know how the stream velocity is varying it ux also we know it.

So these boundary condition need not to be the upstream boundary condition need not to apply it. So by solving these equations today is very easy because you have a very high computing techniques systems with us and there are lot of numerical techniques are developed it to approximate even if Navier-Stokes equations. So if you look at the boundary layers, equations are not that difficult to solve today as compared to 120 years back. So if you look at these are the boundary conditions and that is the solve we can do with a time matching scheme. So more details I am not going it.

I am just encouraging you to follow computational fluid dynamics techniques. Then you can understand it how easily we can solve these two equations to get how the boundary layer thickness are there, how the velocity distributions are going on. So you can get it and also the pressure distributions and wall shear stress and screen fictions. That is today is possible because solving the parabolic equations is not difficult with a numerical techniques what is available to us. Now as the syllabus part, let us go for a solutions, numerical solutions of the laminar boundary layers okay.

So basically we are talking about laminar boundary layers. So to getting these solutions we are just looking at some simplifications as we did it for earlier case that we are looking at the uniform speed flow v is going that is what is the free stream velocity and if you look at the sketch the same and very thin plates okay this is thin plate thin plate because this is to be simplify the conditions that we can get a solutions for this. And basically we are looking at what could be the boundary layer thickness if the free stream velocity is u okay. And considering it is that the boundary layers whatever is here it remains as a laminar boundary layers that means it is a lesser than the critical Reynolds numbers which earlier I said it should be less than 1 log 10 to the power 5 okay. So, if this is the conditions we consider the basic properties like density, the dynamic viscosities and the kinematic viscosities are fairly constants for these regions.

If I do that with these approximations okay that is what we are not going details but So that is the very basic concept is here that is what is called if you consider the outer regions that is what is make it like you have a boundary layer formations this is the y directions and you have x. So basically you are considering it is the outsider regions the velocity of u what is varying it v is a constant. okay is a constant that means this is the cases what we are looking it if you look it the conditions what we are looking it very simplified cases like we have a very thin plates and the boundary layer formations are happening it and we also consider is laminar boundary layers for that if you look at this free streams velocities above just outwards of these boundary layers we can consider this varying with the x but with a simplifies that 0 pressure gradient conditions that is means we are not considering the pressure gradient is a major part for boundary layer formations then the ux will be the v which will be the constant. if that is the conditions the same x momentum equations you can now the simplified it. X momentum equations we can simplified it as these ux components become constants that is what is the graphically representing it.

Case of ux equal to the v that is what is uniform velocity with a boundary layers how we are considering this is the constants values just out of the boundary layers. Now as we have a zero pressure gradient concept as a simplifications we have done it. So that way basically the x momentum equations is further simplified it as a convective terms okay is equal to kinematic viscosities del square u by del y square. So this is further simplified with continuity equations is del u by del x del v by del y. So these the equations need to be solved as I told earlier that for a simplified geometry like this is our domain in these two main we need to solve.

these two equations to know within the boundary layers how the velocity distributions are there, how skin frictions are there. So to solve this, we have a two boundary conditions. As I said it, along these layers, we have the boundary conditions u and v is equal to 0 and this is the velocity distributions we know it and when y equal tends to the infinity and v equal to 0, or u equal to y for all the y act x equal to 0. So that is the reasons where is the irrotational trees there. The Blasius as I mentioned it earlier in 1908 which was a PhD student of Prandtl who is very well known in tolerance studies, okay.

He introduced a similarity variables. I am not going more details to solve these equations with these boundary conditions that is a non-dimensional independent variables in terms of free stream velocity and v and x and the y, the directions in the y directions. So that is the reasons he tried to get it what could be the functions okay using the similarity variables. Those are the concepts earlier used when we did not have a computers to do numerical analysis. The Prandtl used to do hand calculations based on wrong equity methods which is today's you know it many of numerical techniques is a very well known techniques.

So but he did it a hand calculations to get these solutions and then he got these equation solvers as if you look it. after doing hand calculations and all, he get it that eta value is equal to 4.91 and that is what if he can rearrange it, he will get it sigma of the boundary layer thickness by x is equal to 4.91 square root of Reynolds number x. So this is for the boundary layer thickness numerically obtained by Vlesic in 1908 okay.

So and sometimes instead of looking 4.91 people also approximate as sigma by x is equal to approximate is 5 by square root of Reynolds number okay. So we know this what is the thickness of boundary layers which is for laminar boundary layers okay. The same way also he estimate it what will be the wall stress okay that is what you know it that what we can get it if you know this velocity gradient mu times at y equal to 0. So once you know this u variabilities so you can differentiate it and mu times find out the slopes at the thin plates okay.

So we can get it what will be the shear stress okay. So that is what we will get it the shear stress value is like this okay. and if you divide by the rho u square then we get local frictions coefficients okay. So local frictions coefficient I just I am writing once again half of rho u scale capital U squares which is coming out to 0.

664 of Reynolds numbers okay. So this is what as damage is a skin friction factors or called a Lucan frictions coefficients that is what is because of this boundary layer force that how much of force acting on this plate that is the interesting to know it drag force and the lift force. So that is what from this boundary layer thickness for then we can get it what will be the wall shear stress and what will be the screen friction coefficients or the local friction coefficients for the boundary laminar boundary flat plate boundary layers okay. Still we are talking about the laminar flat plate boundary. So that is what you can have a understanding. Now if you look at more details if I looking it from the edge of a plate as I go and if I sketch the boundary velocity distributions this is what the boundary layers.

okay this is what the boundary layers if I sketch the velocity distributions you can see they like this. The tangent and this velocity distribution at the y which is a representing us mu times is representing us the wall shear stress. So you can see that the wall shear stress will be more at the front end but edge going back the wall shear stress will be the less that is the understanding should be there. So as the boundary layer formations happening it very initial phases you will have a more wall stresses as go to the downstream. So as you go to downstream your shear stress decreases shear stress decreases.

So as shear stress decreases so you have a more front you will have a shear stresses that is the reasons if you look at the aircrafts. wings we try to design with such a way that it can take it more loads okay at the boundary layers parts okay that is the reasons if you look at our forefathers used to sharpening the arrows okay. So that surfing of arrows the basically you can understand it how the boundary layer formations happens it and what could be the strongest materials what we need it at the top of the arrows okay that is the necessary to sustains the boundary layers formations. But the same concept is there you can understand it that at the front end will have a higher shear stress and as you go to downstreams your shear stress decreasing patterns are there. And that is the reasons your the materials what you use it at the front end of a wing airplane wings or arrow or missile structures are the more strength, needs a more strength as compared to the downstream part.

That is the you can correlate it with the field conditions okay. So basically if you look at the understanding of boundary layers, talk about how the variations of wall shear stress or screen factors. Now if you look it then another concept what we look it called a displacement thickness. The basically as we know it the boundary layer thickness developments happens it okay. The regions which is closer to that boundary layer formations are happening it.

So the basic idea is comes it as the boundary layer thickness happens it and if I take a just a free out streamlines okay just of a free streamlines okay which I can consider this streamlines without boundary layer formations okay that is the without the boundary layer formations the streamline could go on like this S1 conditions. without boundary layer formations or we are saying that there is no existence of this flat plate okay. So the streamlines which is uniform streamlines what is coming it that can go as just a parallel to this okay. That is why it is a uniform streamflow, uniform streamflow is happening. Now because of the boundary layer formations are there the streamlines will not go straight parallel to that that what will be deflected that what will be deflected.

So you can understand it because whatever the mass flow is going through these two streamlines that should pass through it because the velocity at this point the u is lesser than the capital the free stream velocity. free stream velocity. So considering that that means the distance that a stream line just outside of the boundary layer is deflect away from the wall due to effect of the boundary layers that is what I was explaining it. Because of the formations of boundary layers what is the effect is happening the free stream which is just the outside of the boundary layers how much deflections will happen it.

how much deflected from that. That deflected part is known as displacement thickness. That is what deflected part of the streamline distance that is what we will tell that displacement thickness. So if you look at this is what displacement thickness. So basically this is mass conservation equations concept that we are talking about the mass what will be because of deflecting of these streamlines will be delta star is a displacement thickness into the u that is what will be 0 to y or we can put the infinities okay. So mathematically to put it the infinity some of the books to sell it that we do not know this where it is okay that is the reasons we put it 0 to u.

This is what the deficit of the mass into dy okay. So I have not multiplied the rho I have not multiplied the unit depth okay that is not necessarily. So that way this is what the mass conservations mass component which is part pass through because of the deflected of this streamline. This is what the mass deficit it happens within the boundary layers because the velocity of the boundary layers are lesser than the u value. That is how much of mass deficit is happening it.

So if you just rearrange this time, you will get this part 0 to infinity, okay because We are technically considering it we do not know the where the boundary layer ends it. So you can integrate it but no meaning after this will be the 0 that value. So we will get 1 minus u by dy. So if you know the velocity distributions you can just integrate it and because this is the free stream velocity you know it just integrate it from this point y equal to 0 to infinity okay. So then you will get the displacement thickness.

There is a reductions of velocity field near the boundary layer because of that the mass conservation properties to be hold good it. So the streamlines which is outer streamlines will be deflected and that deflected part we call the displacement thickness with simple mass conservation equations we can get it what will be the relationship. So if I know the u value then we can always compute it what will be the thickness of displacement thickness. So that means we are getting another thickness which is displacement thickness giving a star here okay which is a functions of again just trying to write it to u2 again the same functions but earlier it is 5 it is just close to one-third of by the Reynolds numbers at the x. The same way we can get it what will be the as we do this velocity distributions so we can get it just doing this integrations we can get it this value the equation format will be the same the trend will be the same okay.

So this is what the thickness of the displacement thickness. Now if you look it what actually helps this displacement thickness is to know it we are creating a if I try to understand it that means we have an actual wall okay that is the reasons we put it this is actual wall. but the displacement thickness is gives is as a apparent walls okay. So where we can apply the flow is irrotational and we can apply this Euler equations. So this is what the velocity distributions very close to the slide width.

So this is what a imaginary apparent wall concept is comes it as equivalent thickness we have to add it to solve this Euler equations in irrotational zones. So the displacement thickness help us to create a apparent layers, apparent wall. So over that we can solve it as a the irrotational zones we can use the Euler equations to solve it. The same way if you look at two dimensional flow which is happening between maybe the pipe flow or maybe between two plates. So you can see that this is the formations of boundary layers okay.

And these are the velocity distributions we can have this velocity distribution and in a core regions you can have the constants values like this. So as you go it so this core flow regions reduce and further we develop a fully develop velocity distributions profiles starting from entry which is uniformly built. That is what as equivalent when you put it this is actual velocity distributions and the core regions. Instead of this we introduce the displacement thickness okay. So this is the displacement thickness part and these displacement thickness outsiders this is Euler equations.

So we can assume it more or less the uniform velocity distributions within this okay. So we consider the apparent layers thickness layers as the regions beyond that it is representing for us the Euler equation solvents. So this is what help us for real conditions to coming to a afferent wall concept between these afferent walls we can consider is a flow is as equivalent to irrotational flow and we can use simple stream functions and velocity potential functions and the Bernoulli's equations to solve the Euler equations to get it the flow patterns the velocity distribution pattern. displacement thickness give us to develop the apparent wall concept where we can use Euler equations to derive that. Let us come with a very simple problem which is there in Sindel Simbala book.

There is a low speed wind tunnels being designed for calibrating the hot weather wires okay air is at 19 degree so this is the air flow systems and the test section of this wind tunnel is 30 centimeter diameters okay that is what is okay this is 30 centimeters the test sections and 30 centimeters in the length. So the length is given to us. The flow through this test section must be uniform because that means we have to locate what is the laminar boundary layer thickness. It should not have turbulent boundary layers because then we cannot make a test section power. So we will try to make it the test section such a way that laminar boundary conditions prevails as well as you have to look it what is the amount of thickness is coming it which is will be appropriate for a displacement thickness and also boundary layer thickness.

With a speed, tunnel speed 1 8 to 8 meter per hour and the air speed is 4 meter per seconds. So we need to know it that how much will be central line air speed accelerated by the end of the test sections okay. That is the things where I am not going all the steps wise. So we have to flow is steady, incompressible while it is smooth and disturbance vibrations are kept minimum. You just find out the kinematic viscosity of air at the from any tables defined is given in the book of any fluid mechanics books and all.

So estimate what will be the Reynolds number and of this test because test section is 30 centimetres okay. So is a circular test sections. So basically at the initially you will have this 30 centimeter but at the end you will have a boundary layer formations. You have boundary layer formations. At the leading edge there will be no boundary layer formations but at the end we have the boundary layer formations.

What is that thickness? What is the Reynolds numbers at these points? That is what you can compute it. The Reynolds numbers coming about 7.9610 to the power 4 which is lesser than 10 to the power 5. So the flow is laminar that is critical numbers sometimes 5 to 10 5 million 5 lakhs we consider it is but it is okay. So we can consider is the as you consider is a laminar flow happens it.

So now you are computing the displacement thickness just substituting the displacement equations we can get it the displacement is coming in order of close to the 2 mm. Just you look it. I always you demonstrate it, think it, which is not visible. Mostly we ignore that part. but that is what is majorly controls the boundary layers how the drag force or lift force are happening it and how the shear stress distributions are happening it.

But thickness of as you demonstrated by these examples thickness in the order of millimeters the diameter is in terms of centimeters okay. So that way you can understand it how difficult it was to measure the boundary layers in almost 120 years back okay. So now we have a lot of advanced fluid mechanics equipments, we have high computational facilities so we can conduct very detailed experiments. But 120 years back understanding the boundary layer concept derive the solutions of boundary layer equations by hand written hand calculations that is what was quite appreciable. Now we come to the momentum thickness which is not big the concept which is already we talk about shear stress displacement thickness also the wall shear stress.

Same way it is called the momentum thickness that means As you know it we apply the control volume concept. So we will apply the control volume concept to know it what will be the drag force is happening it. So what will be the drag force is happening it on this plate. So that means let me I sketch it the problems. If you look at that same things I am defining to here this is boundary layer thickness.

I am taking this is my control volume this is my control volume okay that is what very beginning I said this is what my control volume. So there will be a force acting on this on this plate for drag force in the x directions drag force in x directions have dx that is what will be act on this And from this I know uniform stream flow at these points I know it as I was telling it that I have the velocity distributions okay. I have velocity distributions then I have the uniform velocity distributions outer of these boundary layers. that is what is represented here. Mass deficit and this is the delta displacement thickness is showing it how much extra mass that these two we equal each other okay above these free streams okay above these free streams.

So the mass flow deficit due to the boundary layers and that is what is mass that net mass is equate that is what I used to derive for the displacement thickness same concept is coming it but here we will use it as we have done for integral approach. So we will equate the momentum flux in flux and out flux and then we will try to find out what will be the drag force. So this is a simple case representation if you take a control volumes and you try to equate at the momentum flux way in and out and try to find out what will be the drag force acting on this plate okay. That is what looking from this as I earlier I derive it the same way I can get it the mass flux and relationship with the displacement thickness that is what it already I derived it. So you can see it now we are applying it as for these control volumes we are applying it some of the force acting is equal to the net momentum outflow flux from these cross sections.

It is a two dimensional cross section so we can define it what will be the cross sections momentum flux is going out. So this is in, this is out. So if you look at that part and just rearranging the drag force components and x that then you will getting a drag force okay. Momentum thickness is nothing else.

It is talking about only the screen fix and factors okay. It is just introduced as a momentum thickness. as equivalent to moment change because of these boundary layers. So it is defined as viscous drag force on the plate per unit width okay that is what is mentioned it okay which is equal to rho u square theta okay that is as a equivalent representations. That is what if you look it for drag force viscous drag force which F dx by per unit width w and that is what is the momentum flux net what you and you are equating by rho u square okay that is the momentum thickness and you can have a simple calculations then you can get it very simple way the you can remember it 0 to y okay is equal to small u and by capital U this is a free stream velocity. So this is momentum so you can see this multiplications with a u okay that is what is a momentum thickness and many of the times we why we represent as infinities and mathematical point of view as I told it earlier.

So we define it as the momentum thickness. So we can get it what will be the momentum thickness which is a as equal to the skin friction factors. That is what you can understand it because we derived it considering a control volume or you can derive from the velocity distribution both are the same. So but we have the different naming the momentum thickness or skin factors okay. friction factors okay but the both are the same. So that is not a as theta is the same as the delta delta but with a different constants values okay if you can say that 0.

66 upon by r and which is a 13% at the locations x. So mostly this type of concepts comes in a gate or engineering service but you try to understand it as equivalent of thickness we are representing for displacement thickness as well as the momentum flux thickness. Because the change of the flux what is the net force acting as a skin friction factors on a flat plate due to the formations of boundary layers. That is the point we are talking about. Now we are not going to more details. When you go for the turbulent boundary layers just conceptually I will discuss with you.

It is a very complex equations it comes out to be. So the turbulent boundary layers basically we do not get it hand calculations based even if numerical techniques to solve these problems because as I introduced very simple way to the turbulent boundaries. chaos flow with a fluctuating part, the time average components are there. So instead of that the people just try to look at the standard formulas okay which can enough to give the velocity distributions in the boundary layer regions, the turbulent boundary. So one of the very basic equation is that we call this time average components okay. I will show it basically empirical natures that means mostly we are trying to get it from experiments and try to fit the equations and find out the equations okay.

Which is more famous equation for the time average velocity profile is 1 7th power law okay. That means u the velocity by the free stream velocity again I am just sketching it okay you have a the velocity distributions for the turbulent flow and you are just looking it what is this distributions okay which follows this is y directions. which follows approximately okay again approximately 1 seventh power low okay just non-dimensionally it just look at these ones okay. This is the empirical equations and we can try to understand it that when y will be greater than boundary layer thickness then your the velocity ratio becomes this non-dimensional way the velocity distributions is defined by the one-seventh power law okay. So I will show it that so the basically we are trying to look at the turbulent layers and you try as I introduced earlier it will have a large eddy formations and there is lot of mixing happens it and really very interesting subject I can say it.

If you are really interested, learn more turbulent flows which last almost 20 to 30 years. We spent a lot of resources by many scientists to solve these turbulent flows. So you can really get knowledge on the turbulent flow but in this undergraduate course we are not going details. So basically one very simple empirical equations obtained from the experimental data is called is one seventh power law. The same way the equations what give us different formats we will go for next but if you look at that when you have a the shear stress okay to estimating the wall shear stress okay which corresponding to the high screen friction along the surface okay compared to the laminar boundary weight because if you draw the velocity distributions so you will have the high wall stress in turbulent zone as compared to the laminar flow.

So that was high shear stress development will happen it. This is showing it the comparison tables okay only for the smooth plate, boundary layer thickness, displacement thickness, the momentum thickness and local friction coefficients which we have derived for laminar flow okay. So as I said it momentum thickness and local skin friction coefficient both are the same okay that is only we define in different way. If you look at the turbulent flow you can see that how the thickness the thickness will be quite less as compared to the boundary layer laminar flow. But if you look at the value of local skin factors or the momentum thickness is much larger okay.

This is what you can understand from this okay. It will be much larger. That is the same way this is the 1 7th power law because but this law does not hold good for all the cases. They use combined with empirical for the smooth pipes. again they changed the relationship between boundary layer thickness, displacement thickness, momentum thickness and local screen friction coefficients. If you look at that in the turbulence things lot of studies were conducted to get it the best approximations of boundary layer thickness and these properties in turbulent zones.

So these are all can be said is approximations. So similar way there is a log law okay which establish the relationship non-dimensional way relationship between here if you look at the u star subscript star is a friction velocity. okay it is not a velocity it is as equivalent to the dimension of the velocity but it is a representing about the tau is a the bed shear stress the wall shear stress by the rho it is a no dimensionally it is looking the velocity it is but due to the frictions. So that is the reasons with a friction velocity and that is what the velocity distributions not the 1 by 7th power laws but log law concept has come it with having the a, b are two constants which is you can varies from 0.

4 to 0.41 and b varies this. So we get it the log distributions of velocity in boundary layer regions will give us a value of y u star. by kinematic viscosity plus p and here k and b are the constants. I am not going more detailed about what is the constants name and all but you can see that the log law can be used. This is the instantaneous velocity distributions.

The time average velocities will come like this. So we can establish the relationship between velocity distributions with the tau not the bed shear stress okay bed shear stress and as a logarithmic functions non-dimensional logarithmic functions we can use it to have the these relationships okay. So same way it has been tried by many I am not going details about splinding clause, wall wake clause okay. are interest on the turbulent flows, I think I can just advise you to take it the advanced level of fluid mechanics book, read the advanced or the courses which really give you interesting fact how these equations are derived, how we have been trying it to make it better way to presence this turbulence flow velocity field okay which is very chaotic or complex that is what we try to do it with empiricalism with some concept like from experimental data. So the equations are the different and they have a really utility for different case of this. With this let me conclude boundary layers approximations as I said it very beginning we are talking about introductory levels and these are the 3 equations as we talk about boundary layer equations displacement and momentum equations and that is what we have derived it and many of gate or engineering service things are comes under these displacement thickness, momentum thickness and some part of how to estimate the boundary layer thickness that is things are there.

With this let me I think my student groups who really have put a lot of efforts to prepare this presentation. Thank you. Thank you.