Chapter 16: Partial Differential Equations – Basic Concepts

Introduction

Partial Differential Equations (PDEs) form the foundation of mathematical modeling in various engineering fields, especially in Civil Engineering. They are used to describe physical phenomena such as heat conduction, fluid flow, stress-strain analysis, and diffusion processes. Unlike ordinary differential equations (ODEs) which involve derivatives with respect to a single independent variable, PDEs involve partial derivatives with respect to multiple variables. This chapter introduces the basic concepts of PDEs, classification, methods of formation, and standard forms, laying the groundwork for further study and application in engineering problems.

16.1 Definition and Notation

A **partial differential equation (PDE)** is an equation that involves the **partial derivatives** of a function of **two or more independent variables**.

Let $u = u(x, y)$ be a function of two independent variables x and y. Then:

- $\frac{\partial u}{\partial x}$: partial derivative of u with respect to x.
- $\frac{\partial^2 u}{\partial x^2}$: second-order partial derivative with respect to x.
- $\frac{\partial^2 u}{\partial x \partial y}$: mixed partial derivative.

Example of a PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

This is **Laplace's Equation**, widely used in potential theory and fluid mechanics.

16.2 Order and Degree of a PDE

- **Order**: The order of the highest derivative present in the PDE.
- **Degree**: The exponent of the highest order derivative after removing any radicals or fractions.

Examples:

1. $\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial y}\right)^3 = 0 \rightarrow$ Order: 2, Degree: 1
2. $\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 0 \rightarrow$ Order: 2, Degree: 2

16.3 Formation of Partial Differential Equations

PDEs can be formed by **eliminating arbitrary constants** or **arbitrary functions** from a given relation.

A. By Eliminating Arbitrary Constants

Let the relation involve constants a, b: Example: $z = ax + by + ab$ Differentiate partially:

- $\frac{\partial z}{\partial x} = a$
- $\frac{\partial z}{\partial y} = b$

Eliminate a, b to get a PDE.

B. By Eliminating Arbitrary Functions

Let the relation be: $z = f(x^2 + y^2)$

Differentiate partially and eliminate the arbitrary function f, or its derivatives f', to form a PDE.

16.4 Classification of Second-Order PDEs

A second-order PDE in two variables can be written as:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + \text{lower order terms} = 0$$

The **discriminant** is given by:

$$D = B^2 - 4AC$$

Classification based on D:

- **Elliptic** if $D < 0$ (e.g., Laplace Equation)
- **Parabolic** if $D = 0$ (e.g., Heat Equation)
- **Hyperbolic** if $D > 0$ (e.g., Wave Equation)

These types correspond to different physical phenomena and determine the nature of their solutions.

16.5 Linear and Nonlinear PDEs

- **Linear PDE**: The dependent variable and all its derivatives appear **linearly**. Example:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

- **Nonlinear PDE**: Involves **nonlinear terms** like products or powers of derivatives. Example:

$$\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial y} = 0$$

16.6 Standard Forms of First-Order PDEs

A **first-order PDE** in two variables can be written as:

$$F(x, y, u, p, q) = 0$$

Where:

- $p = \frac{\partial u}{\partial x}$,
- $q = \frac{\partial u}{\partial y}$

Linear First-Order Equation:

$$a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} = c(x, y)$$

This can be solved using the **method of characteristics**, which reduces the PDE to a system of ODEs.

16.7 Solution of First-Order Linear PDE – Lagrange's Method

The standard form is:

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} = R(x, y, z)$$

Solution: Integrate the **auxiliary equations**:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

This system of ordinary differential equations yields the general solution.

Example:

Given: $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

→ Lagrange's auxiliary equations:

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{0}$$

Solving gives: $x - y = c_1$, $z = c_2$

Hence, the general solution: $z = f(x - y)$

16.8 Types of Solutions of PDEs

1. **Complete Integral**: Contains as many arbitrary constants as the order of the PDE.
2. **General Solution**: Contains arbitrary functions.
3. **Particular Solution**: Obtained by assigning specific values to constants/functions in the general solution.

16.9 Applications in Civil Engineering

- **Stress analysis** in elastic bodies (Navier's Equations)
- **Flow of water** in soils (Laplace Equation for seepage)
- **Heat distribution** in a rod or slab (Heat Equation)
- **Vibration of structures** (Wave Equation)

PDEs are indispensable for modeling such physical processes where both space and time vary.

16.10 Canonical (Standard) Forms of Second-Order PDEs

Transforming a second-order PDE into its **canonical form** makes it easier to solve analytically.

Consider a general second-order PDE in two variables:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + \text{lower order terms} = 0$$

To simplify, we use a **change of variables**: Let $\xi = \xi(x, y)$, $\eta = \eta(x, y)$

Choose these variables such that the PDE becomes:

- **Elliptic**: $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0$
- **Parabolic**: $\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial u}{\partial \eta}$
- **Hyperbolic**: $\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$

This transformation is guided by the discriminant $D = B^2 - 4AC$.

16.11 Method of Separation of Variables

A powerful technique used to solve linear PDEs, especially with boundary/initial conditions.

Idea:

Assume a solution of the form:

$$u(x, t) = X(x)T(t)$$

Substitute into the PDE, divide both sides to separate variables, and solve resulting ODEs independently.

Example:

Heat Equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Assume $u(x, t) = X(x)T(t)$, substituting gives:

$$X(x)\frac{dT}{dt} = \alpha T(t)\frac{d^2 X}{dx^2} \Rightarrow \frac{1}{\alpha T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2 X}{dx^2} = -\lambda$$

This leads to two ordinary differential equations:

- $\frac{dT}{dt} + \alpha\lambda T = 0$
- $\frac{d^2 X}{dx^2} + \lambda X = 0$

Solve both under given boundary conditions.

16.12 Worked Example – Wave Equation

Problem:

Solve the one-dimensional **wave equation**:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0$$

Subject to boundary conditions:

- $u(0, t) = 0$, $u(L, t) = 0$
- Initial conditions: $u(x, 0) = f(x)$, $\frac{\partial u}{\partial t}(x, 0) = g(x)$

Solution:

Assume: $u(x, t) = X(x)T(t)$

Substitute into PDE:

$$X(x)T''(t) = c^2 X''(x)T(t) \Rightarrow \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$$

Solving gives:

- $X(x) = \sin\left(\frac{n\pi x}{L}\right)$, $\lambda = \left(\frac{n\pi}{L}\right)^2$
- $T(t) = A_n \cos(c\lambda t) + B_n \sin(c\lambda t)$

So the complete solution:

$$u(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

Use Fourier series to determine A_n, B_n from initial conditions.

16.13 Common PDEs in Civil Engineering Practice

Equation	**Mathematical Form**	**Application**
Laplace's Equation	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	Steady-state heat flow, seepage analysis
Heat Equation	$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$	Temperature variation in concrete

Equation	Mathematical Form	Application
Wave Equation	$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	Vibrations in beams and structures
Navier-Cauchy Equation	$\mu \nabla^2 \overrightarrow{u} + (\lambda + \mu)\nabla(\nabla \cdot \overrightarrow{u}) = \rho \frac{\partial^2 \overrightarrow{u}}{\partial t^2}$	Elasticity and stress analysis

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16.14 Numerical Methods (Overview)

While analytical solutions exist for ideal problems, **real-world applications** often require **numerical methods**:

- **Finite Difference Method (FDM)**: Approximates derivatives with finite differences.
- **Finite Element Method (FEM)**: Divides domain into elements and applies variational methods.
- **Finite Volume Method (FVM)**: Often used for fluid flow and heat transfer problems.

Civil engineers use software like **ANSYS**, **ABAQUS**, and **MATLAB** for solving large-scale PDEs in structures, fluid mechanics, and geotechnics.
