Fluid Mechanics
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Lec 27: Cauchy's Equation

Good morning all of you. Let us go for today the derivations of Navier-Stokes equations. Before that quasi equations will derive it and which is a really important subject in the fluid mechanics to understand the computational fluid dynamics. I will try my level best to teach this part of these chapters as is as is possible. So for but the mathematically this chapter looks like very complex but I will try to deliver this lectures as simple as possible not going more detail about mathematics that is my idea. And the basically what we are going to derive it before deriving Navier-Stokes equations we are going to derive Cauchy equations which is very important equations when you go for differential form of linear momentum equations.

So basically we are talking about linear momentum equations in differential forms that is what and that equations derive is the Cauchy equations which is we will going to derive it which is the basic foundations of computational fluid dynamics. Today you can Google the computational fluid dynamics and you can see what are the challenging problems we have been solving in this era. So that is what the basic things and So, let us as I said it earlier that we follow the book of Sinzel Simbala book which is having a more interesting derivations as well as a series of illustrations which makes the problem which makes really the problems to understand the fluid mechanics problems. So that is the strength of this book.

And today there are a lot of MIT courseware are there. And I just encourage you to look at that MIT courseware. No doubt they have the mathematical components is more. But if you try to understand the physical concept, you look at Sinzel Simbala book. And intermediate book is Frankman white book which considers mathematical part as well as a series of illustrations and solved example problems.

Let us before going to this, let I give a solution sheet, CFD problems, computational fluid dynamics problems where here we have used a CFD software is called flow 3Ds and we are solving the very basic things as I discussed in my first class is the velocity field, the pressure and the density. That is what we look at. That is what we look at to solve velocity field, the pressure field and the density field. When you have that problems, here we have considered a problems with a bridge pier. And here if you look at how the velocity field changes with the velocity field which has a component of scalar component and in three directions.

Here I am looking just Cartesian coordinate systems. In that I have the uvw, so this is small v. the scalar component in x directions of velocity field which is a functions of space and the time. That is what I try to introduce in the very beginning classes to say that the scalar components having this functions of positions and the time as it is a Cartesian coordinate systems we are defining as x, y, z and the t the same way if you talk about small v which is the scalar components of in the y directions of velocity components also a functions of x, y, z, t. So, these are all these functions we are talking about.

and using this computational foot diameter tools for a complex flow like flow around a bridge spire, we are solving it and what we are getting it how the velocity fields are changing, how the pressure fields are changing. I am not going more details if you can understand it how the velocity field and the pressure fields are changing it which is a functions of space and time we are getting the solutions like that. Now if I go for the next ones if you look it how do we get this computational fluid dimension solutions by solving these equations okay solving these equations which is known as very famous is Navier-Stokes equations. No doubt it has the limitations which we will discuss more details that what type of limitation is there to solve this fluid flow problems okay. But if you talk about this Navier-Stokes equations which is looks like a nonlinear equation formats in a functions nonlinear partial differential equations which have a functions of u, v, w the scalar fields as well as you have pressures, you have the density.

So that is what is the equations when you solve in a Navier-Stokes equations with certain assumptions which I am not telling you to now. We can solve this real life problems to know it how these the velocity fields are changing it okay. How these velocity fields you can see these vectors you can see the vectors. you can see how the grid patterns are there. So, if you look at that the solution of Navier-Stokes equations gives us to get a approximate solutions of these problems that is what is possible today that is what it happened in last 3 decades.

The computational fluid dynamics is a major tools for us to solve many real life fluid, fluid structures, the environmental problems. Now, today I will talk it how we can derive this naviestic equations step by step. Having the knowledge is that we have this velocity field which is a function of the space and the time The density is a function of phase of time. We also hypothesis of continuum hypothesis. That means we can use these functional behaviors to characterize the flow patterns.

Looking that assumptions, if I go for next ones. I will give a two days about the great professors of Kozyk professor which is way back 1857 and all which is long back. It is almost 200 years back the equations which is derived this is Kozyk equations what we will talk about today. I also talk about the trace tensors in the fluid flow. stress tensors in the fluid flow because in a solid mechanics you know about stress tensors okay.

But today let us talk about stress tensor in the fluid flow and we will also talk about for a simplified cases that when you have a infinitely small control volumes. The idea behind it to as I said it that I have the functions of u, v, w is a scalar field functions with a space and time if I want to derive it and make a small infinite small control volumes where these functions are continuous then I can use a basic Taylor equations to solve the problems. That is the very easy concept what we will follow it to consider a infinitely smaller control volume, so that we can apply the Taylor series concept to derive these the components of mass flux and momentum flux. The same equations we can look it in terms of Newton second laws and will give a very information about the interactions level on Navier-Stokes equations. Before going to the fluid flow problems, let me I sketch very simple things like I have the flow like this and I have the upstream flow and is going out like this.

This is the boundary This is a constraint and as this flow is going like this just introducing it how it is different than solid mechanics and the fluid mechanics. So if you look at the very basics way that if I consider these points I will have some sort of velocity distributions like this. velocity distributions like this. If I consider at these points definitely I will have more velocity as compared to this as the area of the flow has reduced here. So that way I will have more the velocity but that is what I can have with the sketching of the velocity.

The very beginning I can sketch the the velocity distributions approximate flow velocity distributions to understand it how the flow happens. That is what we really discuss in integral approach concept. If I know the velocity distributions how what is my expected velocity distributions as no slip conditions and this I can draw this velocity distribution but in differential forms what we do it. we look it let me I have a small particles infinitely small particles and that is what if it is moving it as infinite small particles are moving it as the velocity distribution changes it definitely it deformations happens it there is deformations of the fluid particles may be linear deformations or shear deformations because as these particles are going it as you know it this is the particles are going it that is what will not be the same. Because there is a contraction there is a change of the velocity distributions it will the deformations here.

Again if it travels it, again it will change the deformations. It will have a different deformations. That is what I am just illustrating with one fluid particles. But if I look at a fluid particles which is starting from here that is okay just have a as I was explaining this concept is that just if you look at the first I told about the velocity distributions what is anticipated if the flow is coming and going out you will have a velocity distributions no slip conditions and as it goes you can see that there is a change of the velocities. Because of the change of the velocity field you can see that the fluid particles which are there that what will go for a different degree of different type of deformations linear difference angular deformations that is what is going to happen it that is what is going to have a the deformations as the as the fluid particles are traveling from this to that direction.

Same way if I consider another fluid particles which will also go through the deformations at different thing See you now to try to understand it that if I consider the n number of the fluid particles and if I conceptually track these fluid particles how the deformations are happening it. That is what is giving me a idea that in a fluid flow when fluid is moving through is that there is a different type of deformations are happening it. Now the concept is coming it that that is what we have to conceptually understand it. When you have a flow systems are going on there are different type of deformations are happening it as there is a change of the velocity distribution, change of velocity distributions, change of the pressure distributions and the change of density distributions. And besides that when there is a deformations as you know from the solid mechanics that whenever you apply the load on a solid you will have a stress formations.

The stress nothing but it is a internal resistance force per unit area that is what is the stress. The stress is internal resistance force per unit area that is what we define as a stress sigma. So that is what we define as a the stress that is a resistance as you apply the load in a solid structures you will can see there is a trace internal resistance force per unit area. The same way when there is a fluid flow is happening it there is a stress is happening at the fluid element level. we are going to talk about this stress component.

This the stress component which in case of the solid mechanics it is easy to measure it but in case of the fluid mechanics we have to have a conceptual framework to understand it whenever there is a flow systems there is a stress developments are happening it that is a responsible for changing the deformations of as the fluid particles are moving it you will have a change of the deformations that is what will reflect in terms of stress components that is what will reflect in velocity pressures and the density field. That understanding has to be there in fluid mechanics. Most of the books has represented in terms of a function which we mix it that all these functions are interdependent each other. that is not having a clear cut and cause and effect towards all are interdependent of your velocity field, the pressure field, the density field and the trace field. Now if you look at the same concept we are taking about the continuum hypothesis but here we talk about the fluid mechanics, we talk about the how the stress are happening within the fluid particles levels or at the understanding level.

Now as I tell it that we should talk about trace okay because as I given a simple examples that we always have a whenever you have a boundary and the fluid particles are moving it there is a stress components okay that is what is deformations are happening there is a stress component on that. So if you look it So, many of the mathematically we define the stress as a tensor and we try to define the fluid mechanics in terms of tensors calculations and all. But being a your undergraduate students I am not going to that levels. Let I try to explain it as simple as possible that we have the trace components. Now let me define it what are the trace.

In case of let me look at this figure look at this figure same figures I am sketching again with components. Let I have component of X and I have a Y and I have a Z. This is the Cartesian coordinates and if I have a surface okay if I consider a surface which is a perpendicular to the X axis is a perpendicular to the X axis it is a parallel to the Y Z plane. is a parallel to the yz plane but is a perpendicular to the x directions. So when is that if I define the stress will be the sigma s the first subscripts indicates me it acting on a plane which is having a perpendicularity in the x direction.

the second shows that the directions of the stress components. Any stress tensors we define it as a two sub grids one showing about the directions second sub grids shows about the directions The first subcrete shows that on which plane it acts it if it makes a perpendicular that is which the directions. That is the perpendicular the plane where it is acting it and that perpendicular for or normal vectors follows the x directions that is the plane surface indicators are coming it. So if it is that I have a sigma xx that is what you can see that x is there I can write a sigma xx when I write same things in these directions it is acting on the same plane. So it will be sigma x and y as it is acting on the y directions.

if I these directions I can write sigma xz the same way if I take a surface the x y and that is what is please remember it in terms of subcrete notations that is what is this would not be can have a confusions here because what is a sigma xs what is a sigma xy what is sigma xz. The same way if I consider a plane surface this plane I can write it as here I will put as the functions this is Y plane, this is Z plane. I am considering a plane surface which is parallel to the X Z plane and if I getting it this component. Now, I can ask my students, they can tell it what are these components. So, if I have this, in case of these x planes, we can get it the sigma xs, sigma xy, xz.

Now, if I can ask that to my students who are there in this class, that what will be this? Sigma, yes, this is the y plane. sigma yy what it will be these things yx this will be sigma yz exactly same things if you look it these components are written like this. So now if you try to understand this the stress is nothing else. It is the internal resistance force per unit area. That is what we are defining the trace components.

We are defining the internal resistance force what is happening it. That is what this is a small infinitely small control volumes I have considered it. In that I am defining the stress field. What do you mean by that? I am defining the internal resistance force what is happening it. per unit area because it is easy for me to derive it that is the reasons we talk about the trace.

But indirectly if you try to understand it for these smaller infinite smaller control volume where I have a dx dy and the dz the dimensions over the surface I am defining it what are the internal resistance force are there per unit area that is the reasons I am talking about the stress field please remember that. That means we are talking about the force per unit areas we are making it as a internal flow resistance force is as n number of fluid particles are traveling it the definitely there will be the resistance part that is what is showing it here. The resistance for unit area as a stress components I can for a Cartesian coordinate systems I can define the stress components that means I can have a stress tensors. Let me write it the trace tensors. So, if I write the trace tensors that is what we will commit as sigma is a trace tensors.

I can define as a matrix of 3 by 3 matrix for these smaller control volumes as the components of sigma xx, sigma xy, sigma xz the same way here I can write it yx I can write it will be the zx this way the sigma yy and yz here will be the zy sigma zz. So if you look at this stress tensor is nothing else again I am repeating it internal resistance force per unit area that is the force components. Per unit area we are defining which may have the 9 components the sigma xxx, sigma xy, xz that means it is acting on the plane surface which is perpendicular to x direction that means yz plane in direction of x, y and z direction. the same way it is happening it if I want to try to interpret it in these directions that means all this internal resistance force acting along the x directions they have the component of xx yx and the zx this is what the acting in the x direction this is what acting in the y directions this is what acting on the z direction this is what acting on a y z plane and it has the directions normal to that that is what is happening in this way. So, if you try to look at these equations as a stress tensor as a internal resistance force per unit component that means we are nothing doing great mathematics we are just equating some of the force components.

at infinitely small control volume where we can apply the Taylor series we can apply basic vector calculations nothing else. Looking that let me I go for the next one to tell you what we are going back. That is why please do not have a fear about these termologies and the control volumes. That is what is when you want to write for a differential equations formats in your the control volume have to be infinitely small control volumes that is which is idea. So that is the reasons you can get the velocity field, the pressure field and the density field.

Now if you look it again I am as we discussed we say Reynolds transport theorems applying the Reynolds transport theorems having this p value is momentum the linear momentum equations. This is the linear momentum equations. So mass into velocity that is what when you apply it I will get the same form as I discussed in Reynolds transport theorems. The sum of the force is going to have the rate of change of the momentum flux within the control volume, net outflux of the momentum flux passing through the control surface.

This is what the Reynolds transport theorem. this is what the Reynolds transport theorems as we discuss more details in Reynolds transport concept. Now what is the basic difference we have now? Because when I consider a smaller control problems like this on which there are the trace tensors already I defined it. Trace tensors already we have defined it which is the again I am repeating it is a force per unit area that is what is a force per unit area that what can we tell it this is what the surface force component that is what if I multiply with a stress with area I will get it the internals resistance force components which is acting on the control volumes infinite small control volumes which will give me surface force component besides that I need body force component. which is you know it where you just consider the weight of this infinite small control volume. If it is that let me I write it the force what is acting here will have the body force and the surface force.

So, that is the same things as we discuss in a Reynolds transport theorems the integral approach. Only the things what is different here the body force we are computing as a integral volume integrals of rho g dV, V is the volume here. V star is the volume of this control volumes which is a dx dy dz. So it is a parallel 5.

So you can find out the volumes is will be dx dy dz. That is what is integrating on this body force part. What is the surface force? Very easy things. That is what the integrations of sigma ij into n into da is nothing else as I try to make it that do not have a very afraid it the same equations we are applying for a infinitely small control volumes where there is a trace field as we define it stress into area that is what is a surface force. where you are putting a dot product because we are not assuming this x, y, z plane with a coordinate axis.

So, that is you can have a general format. So, we can get it this is the surface force, this is what the gravity force and g can also sometimes it is better is represent as the vector components. some of the force components we are getting it one is a gravity force that is what is the body force and also getting this what is the surface force. So, now if you look it in the next part what I am just looking it if you look at this control volumes before going to next flat again I will tell it if I take a any stress component sigma xx, please remember it they also the functions of space and time. So, we can use the Taylor series and continuum hypothesis to consider the trace components also as a functions of space and time. now if you look it I have the control surface I am just looking this component how it is coming for a control volumes where is it very large control volumes I can find out this momentum flux and the corrections factor for this as I have convert these control volumes into a infinitely small control volumes.

So, now the control volume which is not a bigger control volumes is infinitely small control volumes as I have a infinitely small control volumes. Now if you look it I have a components in terms of momentum flux components I am looking it that I have a component rho into u is the max flux per unit area that means if I put the unit of each component is kg per meter cube into meter per second that is what gives me kg per meter square and second that means max flux kg per second per meter square is coming it rho into the u. Always you always put the dimensions to check component or understand what is meaning of that. That is what the basic fluid mechanics understanding that you always substitutes the unit to understand what is that. When you have a rho into u that means the density, mass per unit, volumes into velocities meter per second I am getting it mass flux mass per unit time per unit area that is what is mass flux.

If I multiply it rho u into u that is what will give me kg meter square second into meter sorry meter for seconds which is gives is kg m by second square y meter squares just that is what the momentum flux per unit area that is what is the unit of both force kg is a mass into the accelerations per unit area. That is try to understand it this is also a swarm a force components per unit area I am getting it because of the momentum flux is coming from this rho u component is going in this direction. Try to understand it looking this unit of this part the rho u is a mass flux. mass flux kg per second per unit area that is what is coming from this surface what is the momentum flux because of mass flux is coming in in the y directions that what if you multiplied with rho u into u that is what will give it momentum flux per unit area that is what a kg mass into acceleration power that is a force is coming due to the momentum flux is what is going on or going out. Same way if you can write it the rho u u rho u v rho v u rho w u all you can understand it all are the nothing else is a force per unit area happens it because of momentum flux is going in x direction because of max flux of this y direction because of max flux because of max flux in the b direction that is what is representing or max flux representing in the y direction.

So, try to understand it that when you have an infinitely small control volumes your max flux is coming in the three directions. also it has the momentum flux in the ninth directions that that is the sorry in a three dimension. So, if you compute it, it will have a nine components it will have a nine components that is a equivalent a force per unit area. As remember it we talk about the stress is a internal resistance per unit area same way with a rho u u components we can look it and physically interpreted it is nothing else is a momentum flux per unit area as h equivalent force is happening it because of the momentum of the fluid flow is coming into this control volume or going out from this control volume. How much of momentum flux is coming in going out from this control volume.

Now if you look it that if I am to look this component that how if I take a infinite small control volumes I know it my rho uu rho uv rho uw like this I have the nine components as the momentum flux per unit components are there and if I define this is here the functions behaviors at this point rho u u just I am putting in one dimensional things taking this two surface. So, my rho u u is at this point which is functions of space and time if I look at a distance d by x by 2 distance and this is my x directions. using the first Taylor series affirmations what I will get it here at this surface will be rho uu plus because this is a function circuit of first derivative into dx by 2 as the distance is ds by 2 and my x direction is a positive in this direction. If I am to compute here, this will be rho uu minus del by del x rho uu dx by 2. So, for this control volume if I do not look it fully three-dimensional just to simplify it a two-dimensional way that there will be a if there is a rho uu as the functions is a continuous functions at the centroid point the value will be at this point will have a these first two components of the Taylor series will explain me what will be the value at this phase as well as this phase.

And if I look at the net just subtracting this along this the momentum flux per unit area that is what you are doing it per unit area no doubt if you just subtracted it you will get it rho by dx rho uu dx. So, this will be cancelled out. So, you will have a dx by 2 and this you will have a rho u dx if you multiplied with area you will get it force due to the moment of flux that is what per unit area is there you multiply it with area that is what is the area will be the it is acting over the surface of dy and dz. So, if you look it as earlier in the classes we try to simplify these terms. Now we are trying to write these terms in terms of momentum flux as defined by rho uu what will be component here.

also what will be component here what will be component in these directions because of rho v is a max flux similar way rho w is a max flux when it is coming it that momentum flux what is comes in u directions that is what is represented the v rho v is a mass flux what is coming into this which is having a velocity components what is a causing the momentum flux into this control volume that is what is representing and similar is going out. So, if you look it the derivations is a very simple it is not complicating two things only the things are there they assuming it all these functions are continuous functions and they can use the Taylor series. to approximations of this that is what they have done it. And you can get net momentum flux per unit area just subtracted for each directions x directions with a different part. So, more detail you can look it in a book but this is what basic concept.

Now if you look it another component just time to highlighted here that this component we can because it is infinitely small we can approximation to these ones that means again I am just writing it rho v the change of rho u components in dv. that what we can approximate it because a smaller control volumes the dB will be dx dy dz and we can approximate this is rho u. You can approximate it. This is smaller control volumes.

The variations within these control volumes are not that significant. So, you can use this, come out these functions and you can write the answer. So, basically we will try to understand it that how the approximations have done in deriving these quasi equations. Now if you look it as I deriving these things that we can get it the components as I was just deriving for the x directions. So, same way you can directions for the v max flux in the y directions z directions the sum of the momentum flux components sorry the momentum flux components. This is defining as a outflow and inflow the same concept if you can look at this is the same concept that is what will come it del by del x rho u plus rho u into u.

plus same way I am just writing it rho v u rho z w u this is the volume of the control Now you do it what is dy dx of change of these functions along the x direction this is a change of functions in the y direction change of the function in the z direction that is what we are looking it. The momentum flux how it is changing in the x direction that what will be the net force as equivalent force and this is equivalent force as equivalent force components we are getting it by just physically try to interpret it the force components because of net momentum flux are coming it okay the fluid particles are coming in going out that is what you will be coming the momentum flux also going out the momentum flux because of fluid particles are coming into this control volumes also going out of the control volume that components we are deriving it their gradient in x directions y direction z directions showing you are the force components which is acting on this because of the moment of flux what is going it per unit per unit volume that is the reasons dV is there. Now if you look at the next slides what we are looking at that if I apply that same concept for the all these directions. Now if I look at just look for the stress field okay. Again I consider the sigma xx is a stress which is a functions of space and the time and I can use Taylor series to approximate how the sigma xx variabilities will be there.

Same way as I did it for momentum flux. This is nothing else. the same way if this is the at these centers I have the sigma xx what will happen it a surface which is a dx by 2 distance that is what will come with this part multiply it with area that is what will give a force component same way what is the force component on these directions I can find out on this surface and this surface multiply this. Same way you can have a sigma yx, sigma yx component, sigma zx component. That what we can look it. I told it the sigma xx, sigma yx, sigma zx all are acting in the x directions. over these control volumes all are acting on the on the x axis directions we are just looking it the how what is the force is happening it because of along these x directions because of sigma xx, sigma yx, sigma zx as their values we can approximate it based on the Taylor series then we can always estimate it that in a x surface directions this is what will be the gradient that is like moment of flux.

The same way the moment of flux we are going to get the gradient in three directions will be representing me that what is the surface force is going to act it. That means the surface force is acting in I just write it again will give us sigma xx y sigma yx plus sigma z sigma zx into dx dy dz that is the components we will get it. So, you can just understand it that how the same concept of the Taylor series same concept of the functions variabilities as the stress variability if I consider it then I can estimate it what is the force is going acting on this surface level of this control volume. as we have computed for the momentum flux and the surface force components and the gravity force components. Now it is a quite easy for us to write a basic equations that it comes like this in x momentum directions the equations comes like this if I equate the components of I am getting for this the body force the surface force components.

This is the net out flux of the momentum flux that is what in the x directions I am getting on this per unit volume. This is the net out flux change of the momentum momentum flux within this control volume with respect to time that is what is showing it this. So in a partial differential equations format I can get it this all these components just equating the momentum flux the change of the storage of a momentum flux and this is the body force and the surface force. So if you look it as we did it for integral concept using Reynolds transport theorems the same we have now we are looking it in a differential equations format. The derivations is same it has a certain assumptions as I am trying to explaining you with that these are all the functions and the Taylor series with appropriate control volumes if you consider it.

then we will get it this x momentum equations like this. So, if you can understand is all are the force components and that is what is we are splitting way we are looking the force per unit volumes of all these terms. Now in the y directions and z directions we can get the equations for much like this. So, this is for y directions, this is for z directions. So, we can have now the three equations from Reynolds transport theorems as we are applying for x directions and y directions z directions, I can have the three equations representing this momentum flux in this component and this component. This is because of trace field, this because of the body force component, this is because of momentum flux and this is what because of change of momentum.

storage within the control volume with respect to time. That is the summation part what we are doing it. Now if I look at a very basic vector form which is which can easily interpreted it and I can write it that equations format. Because it is easy to remember the vector as compared to these three equations you have into rho g is a vector into sigma ij. this is components you can just have a dot product the you can get it stress Q this is a body force component again I am repeating it this non-linear terms also you can expand it just have a this rho v v and change of the momentum flux within the control volumes which with respect to time that is what is given it this is what Cauchy equations which is derived almost 200 years back. I think with this let us conclude. Today we will conclude this. Thank you.