Chapter 13: Convolution Theorem

13.1 Introduction

The **Convolution Theorem** is a powerful result in the theory of Fourier Transforms and Laplace Transforms. It simplifies the process of evaluating the transform of a product of two functions. In the context of engineering, especially Civil Engineering, it is particularly useful for solving linear systems, integral equations, and differential equations encountered in structural analysis, fluid flow, heat transfer, and vibration problems.

Before diving into the theorem itself, it's important to understand the idea of **convolution**, its definition, and how it interacts with transformation techniques such as the Laplace Transform and Fourier Transform.

13.2 Definition of Convolution

Let $f(t)$ and $g(t)$ be two piecewise continuous functions defined for $t \geq 0$. The **convolution** of f and g, denoted by $(f * g)(t)$, is defined as:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

This definition is symmetric in f and g, meaning:

$$(f * g)(t) = (g * f)(t)$$

Interpretation:

Convolution blends two functions such that one is flipped and shifted across the other. In physical terms, it describes how the shape of one function is modified by another — a concept widely applicable in engineering systems analysis.

13.3 Convolution Theorem for Laplace Transforms

If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, then the Laplace transform of their convolution is:

$$\mathcal{L}\{(f * g)(t)\} = F(s) \cdot G(s)$$

Proof Outline:

1. Use the definition of convolution.
2. Apply Laplace Transform to the convolution integral:

$$\mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\}$$

3. Change the order of integration (by Fubini's Theorem or substitution).
4. Evaluate the inner integral and show it equals the product $F(s) \cdot G(s)$.

Thus, convolution in the time domain becomes multiplication in the Laplace domain.

13.4 Convolution Theorem for Fourier Transforms

Let $f(t), g(t) \in L^1(\mathbb{R})$, and let their Fourier transforms be $F(\omega)$ and $G(\omega)$. Then:

$$\mathcal{F}\{f * g\}(\omega) = F(\omega) \cdot G(\omega)$$

And conversely:

$$\mathcal{F}^{-1}\{F(\omega) \cdot G(\omega)\}(t) = (f * g)(t)$$

This is extremely useful when analyzing frequency-domain behavior of physical systems.

13.5 Properties of Convolution

1. Commutative Property:

$$f * g = g * f$$

2. Associative Property:

$$(f * g) * h = f * (g * h)$$

3. Distributive over Addition:

$$f * (g + h) = f * g + f * h$$

4. Identity Element:

The Dirac delta function $\delta(t)$ acts as the identity:

$$f * \delta = f$$

These properties make convolution a fundamental operation in linear time-invariant (LTI) systems.

13.6 Applications in Civil Engineering

1. Structural Analysis:

Convolution can model how structures respond over time to varying loads — crucial in earthquake analysis and dynamic loading conditions.

2. Heat Transfer:

Temperature distribution in slabs or columns subject to variable heat sources can be expressed as a convolution of the input (heat function) with the system's impulse response.

3. Groundwater Flow:

In hydrology, convolution helps solve flow problems in porous media, especially in estimating response functions to precipitation input.

4. Vibrations of Beams and Plates:

The response of damped vibrating systems (common in bridges and buildings) to external forces is found using convolution with Green's function or impulse responses.

13.7 Solving Differential Equations Using Convolution

Consider a second-order linear ordinary differential equation with initial conditions:

$$y'' + ay' + by = f(t)$$

Taking Laplace Transform and applying initial conditions:

$$s^2Y(s) + asY(s) + bY(s) = F(s) + \text{initial terms}$$

Solving for $Y(s)$:

$$Y(s) = \frac{F(s)}{(s^2 + as + b)} + \text{terms from initial conditions}$$

Now, using convolution:

$$y(t) = (f * h)(t)$$

Where $h(t)$ is the inverse Laplace Transform of $\frac{1}{s^2+as+b}$, which acts as the **impulse response**.

13.8 Evaluation Techniques for Convolution Integrals

Method 1: Direct Integration

Applicable when both $f(t)$ and $g(t)$ are piecewise defined and manageable:

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau$$

Method 2: Using Laplace Transforms

$$\text{Step 1: } \mathcal{L}\{f(t)\} = F(s), \quad \mathcal{L}\{g(t)\} = G(s)$$

$$\text{Step 2: } Y(s) = F(s) \cdot G(s)$$

$$\text{Step 3: } y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

This approach simplifies many problems in signal processing and systems analysis.

13.9 Examples

Example 1:

Evaluate the convolution $(f * g)(t)$, where:

$$f(t) = t, \quad g(t) = e^{-t}$$

Solution:

$$(f * g)(t) = \int_0^t \tau \cdot e^{-(t-\tau)} d\tau = e^{-t} \int_0^t \tau e^{\tau} d\tau$$

Using integration by parts:

$$= e^{-t} \left[(t-1)e^t + 1\right] = (t-1) + e^{-t}$$

Example 2:

Solve $y'' + y = \sin t$, with $y(0) = 0, y'(0) = 0$, using convolution.

Solution:

Take Laplace Transform:

$$s^2 Y(s) + Y(s) = \frac{1}{s^2+1} \Rightarrow Y(s) = \frac{1}{(s^2+1)^2}$$

Now find $y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

Using known inverse transform:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = t \cdot \sin t$$

Thus, $y(t) = t \sin t$

13.10 Graphical Interpretation of Convolution

Understanding convolution through a graphical lens is crucial, especially for interpreting system behavior in engineering.

Steps to Graphically Compute $(f * g)(t)$:

1. **Flip $g(\tau)$ to get $g(-\tau)$.**
2. **Shift $g(-\tau)$ by t to obtain $g(t-\tau)$.**
3. **Multiply** $f(\tau) \cdot g(t-\tau)$ for all $\tau \in [0, t]$**.
4. **Integrate the product over τ from 0 to t.**

This process essentially "slides" one function over another, multiplying and summing their overlapping parts at each moment t.

Use in Civil Engineering Contexts:

- For time-dependent structural load analysis, convolution can visually represent how a force applied at one point in time influences the system at another.
- In **Finite Element Analysis (FEA)**, convolution allows the use of **impulse response functions** to build the full solution over time.

13.11 Example 3: Piecewise Convolution

Let:

$$f(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & t > 1 \end{cases}, \quad g(t) = t$$

Find $(f * g)(t)$.

Solution:

We compute:

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau) \, d\tau$$

Case 1: $0 \leq t \leq 1$

Since $f(\tau) = 1$ for $\tau \in [0, t]$:

$$(f * g)(t) = \int_0^t (t - \tau) \, d\tau = \left[t\tau - \frac{\tau^2}{2} \right]_0^t = t^2 - \frac{t^2}{2} = \frac{t^2}{2}$$

Case 2: $t > 1$

Now $f(\tau) = 1$ only from $\tau = 0$ to 1:

$$(f * g)(t) = \int_0^1 (t - \tau) \, d\tau = \left[t\tau - \frac{\tau^2}{2} \right]_0^1 = t - \frac{1}{2}$$

Thus:

$$(f * g)(t) = \begin{cases} \frac{t^2}{2}, & 0 \leq t \leq 1 \\ t - \frac{1}{2}, & t > 1 \end{cases}$$

13.12 Convolution in Discrete-Time Systems (Digital Civil Systems)

While most traditional civil engineering systems are modeled using continuous-time functions, **modern civil infrastructure**, such as smart monitoring systems, sensors in smart bridges, or automated irrigation systems, require **discrete convolution**.

Definition:

For discrete functions $f[n]$ and $g[n]$, the convolution is defined as:

$$(f * g)[n] = \sum_{k=0}^{n} f[k] \cdot g[n-k]$$

Applications:

- **Digital Signal Processing (DSP)** for vibration data from sensors.
- **Load data analysis** from building management systems.
- **Automated construction systems** responding to control signals.

13.13 Example 4: Discrete-Time Convolution

Given:

$$f[n] = \{1, 2, 1\}, \quad g[n] = \{1, 1\}$$

Find $(f * g)[n]$.

Solution:

We compute the convolution for each n:

- $n = 0$: $f[0]g[0] = 1 \cdot 1 = 1$
- $n = 1$: $f[0]g[1] + f[1]g[0] = 1 \cdot 1 + 2 \cdot 1 = 3$
- $n = 2$: $f[0]g[2] + f[1]g[1] + f[2]g[0] = 0 + 2 \cdot 1 + 1 \cdot 1 = 3$
- $n = 3$: $f[1]g[2] + f[2]g[1] = 0 + 1 \cdot 1 = 1$

Hence, $(f * g)[n] = \{1, 3, 3, 1\}$

13.14 Convolution in Green's Function Method

In solving differential equations in civil systems (beams, plates, or soils), the **Green's function** $G(t, \tau)$ gives the response at time t due to an impulse at τ.

The solution to a system with input $f(t)$ is given by:

$$y(t) = \int_0^t G(t, \tau) f(\tau) d\tau$$

This is a convolution integral: $y(t) = (G * f)(t)$

Use Case:

- **Soil settlement under time-dependent loading**
- **Bridge deflection under moving loads**

13.15 Civil Engineering Case Example: Convolution in Structural Dynamics

A building subjected to ground motion $f(t)$ due to an earthquake has a known **impulse response function** $h(t)$. The **displacement response** $y(t)$ of the building is given by:

$$y(t) = (f * h)(t) = \int_0^t f(\tau) h(t - \tau) d\tau$$

If:

- $f(t) = e^{-t} \sin t$: ground motion
- $h(t) = \frac{1}{m\omega} \sin(\omega t) e^{-\zeta t}$: system response

Then using convolution, you can determine how the building will behave over time during and after the earthquake.

This approach is the foundation of **response spectrum analysis** in seismic design.
