Solid Mechanics

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Lecture - 27

Theory of Beams

Abstract

Hello everyone! Welcome to Lecture 27! In this lecture, we will learn about the theory of beams.

1 Introduction (start time: 00:38)

A beam is a slender body whose two dimensions are very small compared to the third dimension. It is geometrically characterized by its length and cross-section. A beam is also characterized by its aspect ratio or slenderness ratio, i.e., the ratio of its length to a representative dimension (such as diameter) of the cross-section. The aspect ratio of a beam is usually of the order of 10 or higher. The body shown in Figure 1 is not a beam since it does not have a well-defined length or cross-section whereas the body shown in Figure 2 qualifies to be a beam.

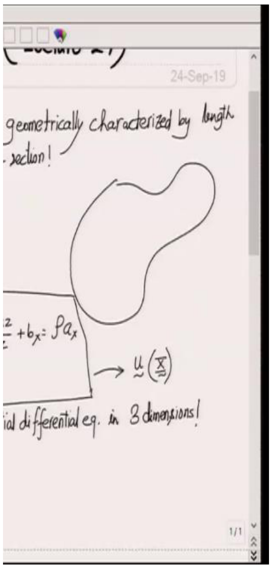

Figure 1: A general three-dimensional body which cannot be classified as a beam.

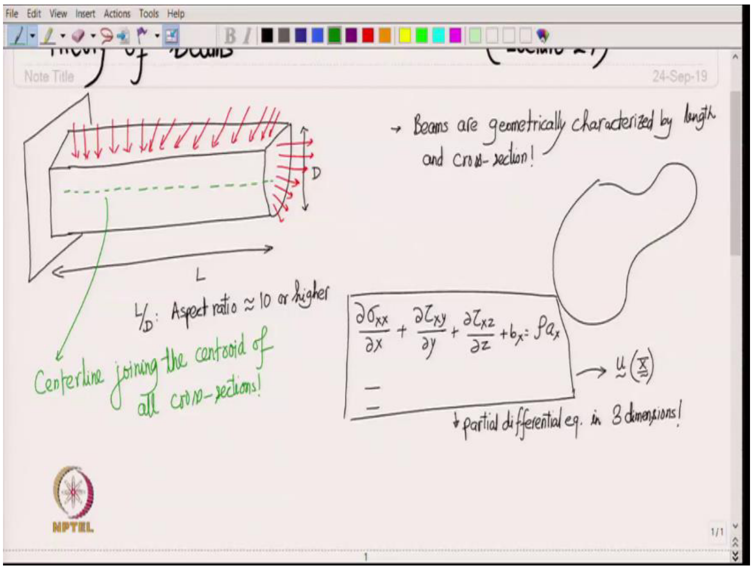

Figure 2: A beam (before deformation) clamped at one end and subjected to arbitrary loading.

When applying load (e.g., terminal force/moment, distributed load etc. as shown in Figure 2) to a beam, we want to know how the beam deflects. The beam in Figure 2 has some general traction acting on its top and right surfaces. If we want to solve for the deformation of this beam, we can always use the following three-dimensional stress equilibrium equations:

$$\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+b_x &= \rho a_x\\ \frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+b_y &= \rho a_y\\ \frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{zy}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}+b_z &= \rho a_z\end{aligned} \tag{1}$$

which is a system of partial differential equations in three dimensions. We need to further plug in the stress-strain relation and the strain-displacement relations to finally obtain the equations in terms of the displacement ($\underline{u}(\underline{X})$) of every point of the three-dimensional beam. It is usually impossible to solve such equations using pen and paper for an arbitrarily shaped body or even for a beam having arbitrary cross-section. We need to resort to numerical computation techniques. In beam theory, we look for an alternate and easier method to find the deformation of beams. The goal is not to find the displacement of every point of the beam but just its centerline which is typically the line joining the centroid of all the cross-sections. If we draw the exact deformed beam subjected to arbitrary loading, it might look as shown in Figure 3a: the displacement of every point of the beam is required to draw it. But, if we can accurately find just the deformed centerline and neglect the displacement of other points of the cross-section, we would get the deformed beam as shown in Figure 3b: all the cross-sections which were initially planar are constrained to remain planar even after deformation. In fact, all the cross-sections are kept rigid and just drawn about the deformed centerline: the envelope of these cross-sections forms the approximate deformed beam.

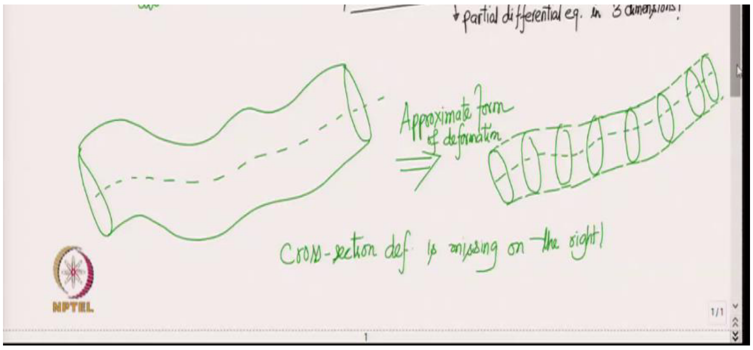

Figure 3: (a) The exact deformed configuration of a beam (b) an approximate deformed configuration of the same beam

It turns out that if the beam is slender enough (i.e., aspect ratio is greater than 10), the cross-section deformation is indeed negligible and the approximate deformed beam is then an excellent approximation of the actual deformed beam. Thus, for deformation problems involving slender beams, we can just solve for the its centerline and put the cross-sections rigidly about the centerline to construct the entire three-dimensional deformed beam. This is the essence of beam theory.

Let us consider the placement of the rigid cross-sections more closely. When we have obtained the deformed centerline, there are various ways to place the rigid cross-sections on that. The cross-section can be kept such that its plane normals are oriented arbitrarily as shown in Figure 4a or we can align the cross-section normals with the deformed centerline tangents as shown in Figure 4b.

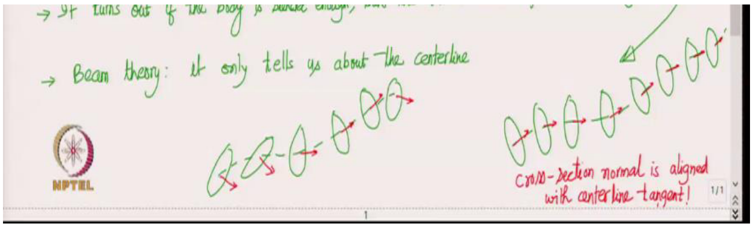

Figure 4: (a) Cross-section normals are oriented arbitrarily relative to the centerline tangents (b) Cross-section normals are aligned with centerline tangents

Thus, a beam theory, in general, solves for the centerline and the cross-section orientation. It turns out that the equations required to get these two variables are simple ordinary differential equations which are much easier to solve than the three-dimensional partial differential equations in (1). In fact, it would also be possible to obtain analytical solutions to the ODEs of beam theory in many cases.

2 Euler-Bernouli Beam Theory (start time: 14:44)

In this theory, it is assumed that the centerline tangent and the cross-section normal are aligned (see Figure 4b). To get the required equations, we can invoke the following bending formula derived in an earlier lecture:

$$M = EI\kappa. \tag{2}$$

Here, M represents bending moment, EI bending stiffness and κ represents bending curvature. We also know from differential geometry that the curvature κ of a curve can be written as

$$\kappa = \frac{\frac{d^2y}{dx^2}}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}} \tag{3}$$

where (x,y) denote the coordinates of a point on the curve. We can apply the above formula to the beam's centerline with y denoting its transverse deflection (see Figure 5a) from the initial straight configuration.[1] The inverse of curvature also equals the radius of the best-fit circle to a curve as shown in Figure 5b.

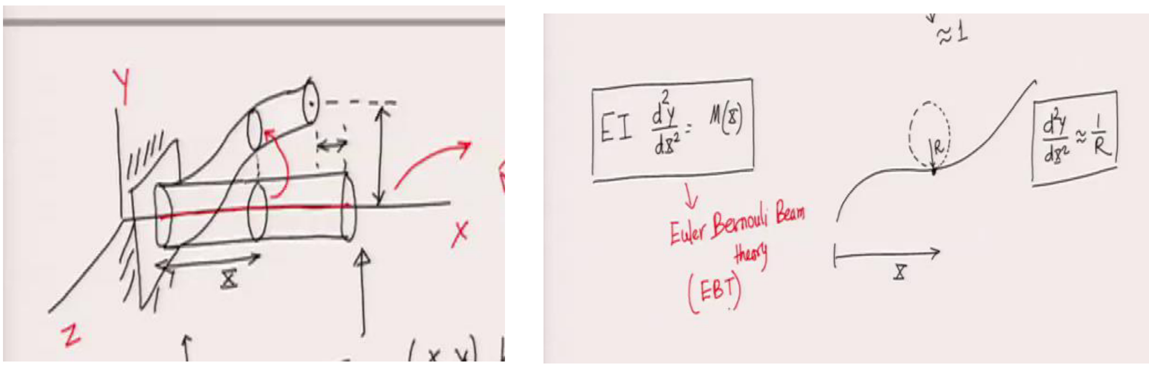

Figure 5: (a) An initially straight beam deforms due to application of some load (b) deformed centerline shown together with the local radius of curvature

The above equation is nonlinear in the unknown deflection y where the axial coordinate x is also an unknown. In the Euler-Bernouli beam theory, the axial displacement is also assumed to be negligible. This means that the deformed axial coordinate x and the undeformed axial coordinate X are approximately the same, i.e.,

$$x \approx X. \tag{4}$$

The transverse deflection y can also be thought of as being the function of the undeformed axial coordinate X. Accordingly, the curvature formula can be approximated by

$$\kappa(X) \approx \frac{\frac{d^2y}{dX^2}}{\left(1 + \left(\frac{dy}{dX}\right)^2\right)^{\frac{3}{2}}} \tag{5}$$

To linearize the above expression in deflection y, it is further assumed that the magnitude of the slope of the centerline is very small, i.e.,

$$\left|\frac{dy}{dX}\right| \approx 0 \;. \tag{6}$$

using which equation (5) becomes

[1] We are restricting to planar deformation of the beam. Hence, the centerline remains in x–y plane after deformation.

$$\kappa(X) \approx \frac{d^2y}{dX^2} \tag{7}$$

The curvature now becomes linear in the transverse deflection. Substituting the above relation in equation (2), we get

$$\boxed{EI\,\frac{d^2y}{dX^2} = M(X)} \tag{8}$$

which is the governing equation of the Euler-Bernouli beam theory (EBT). We can solve the equation to obtain the deflection of the beam. To summarize, the following assumptions are made in deriving the equations of Euler-Bernouli beam theory:

1. cross-section normal and centerline tangent are aligned
2. axial displacement of the beam is negligible
3. slope of the beam's centerline is negligible

2.1 Example 1 (start time: 26:32)

Suppose we have a straight beam which is clamped at one end and is subjected to a transverse load *P* at its other end as shown in Figure 6.

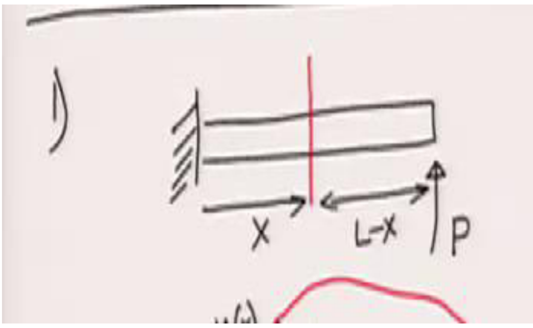

Figure 6: A beam clamped at one end. A transverse load *P* is applied on the free end.

Let us obtain the deflection of the beam using the Euler-Bernouli Theory (EBT). We first need to find the bending moment profile (i.e., bending moment M as a function of *X*). Let us cut a section in the beam at a distance *X* from the clamped end. The free body diagram of the resulting two parts of the beam are shown in Figure 7.

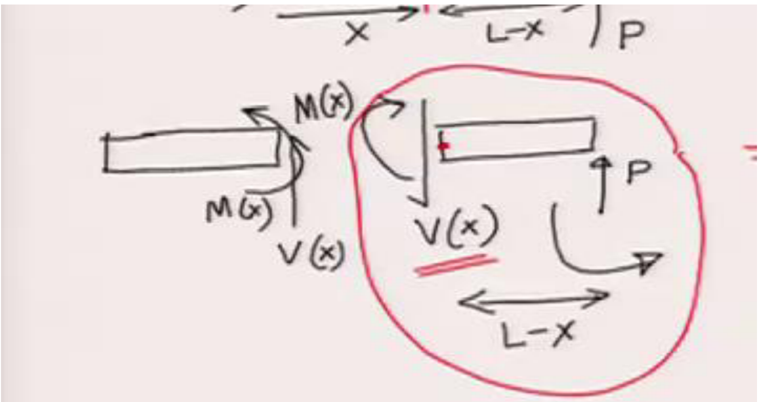

Figure 7: Free body diagrams of the left and right parts of the original beam shown in Figure 6.

For the left part of the beam, at its right end, the cross-section normal points in $+x$ direction. Therefore, here the shear force *V(X)* will act upwards and the bending moment *M(X)* will act in anticlockwise direction. The reactive shear force and reactive bending moment from the clamped support are also shown at the left end of this part. If we consider the right part of the beam, external force *P* acts at its rightmost end. As the cross-section normal of its left end is in $-x$ direction, the shear force *V(X)* acts on it in $-y$ direction. Similarly, the bending moment *M(X)* at the left end acts in clockwise direction as per convention. Moment balance for the right part of the beam about the centroid of its left-end cross-section results in

$$-M(X) + P(L - X) = 0 \Longrightarrow M(X) = P(L - X). \tag{9}$$

Plugging this expression in equation (8), we get

$$\frac{d^2y}{dX^2} = \frac{P}{EI}(L - X) \tag{10}$$

This is a second order linear differential equation. So, we need two boundary conditions to solve it. If there were additional unknown parameter in the equation, we would have required more boundary conditions to find them. At the clamped end ($X = 0$), the centerline cannot deflect and the cross-section cannot rotate either. As the cross-section normal has to be aligned with the centerline tangent in this theory, the slope of the centerline itself will give us the cross-section normal. Thus, we get the following two boundary conditions:

$$y(0) = 0, \tag{11}$$

$$\frac{dy}{dX}(0) = 0 \tag{12}$$

Integrating equation (10), we obtain

$$\frac{dy}{dX} = \frac{P}{EI}\left(LX - \frac{X^2}{2}\right) + C_1 \tag{13}$$

The constant C_1 turns out to be zero upon applying the boundary condition (12). Integrating further, we get

$$y = \frac{P}{EI}\left(L\frac{X^2}{2} - \frac{X^3}{6}\right) + C_2 \tag{14}$$

The constant C_2 also turn out to be zero upon using the other boundary condition, i.e., equation (11). Thus, we get

$$y = \frac{P}{EI}\left(L\frac{X^2}{2} - \frac{X^3}{6}\right) = \frac{PL^3}{6}\left(3\left(\frac{X}{L}\right)^2 - \left(\frac{X}{L}\right)^3\right) \tag{15}$$

The tip deflection can be obtained by substituting $X = L$ in the above equation which yields

$$y^{tip} = y(L) = \frac{PL^3}{3EI} \tag{16}$$

Thus, larger the force, larger is the tip deflection. Also, larger is the bending stiffness EI, smaller is the deflection.

2.2 Example 2 (start time: 34:02)

In this example, we think of a simply supported beam as shown in Figure 8.

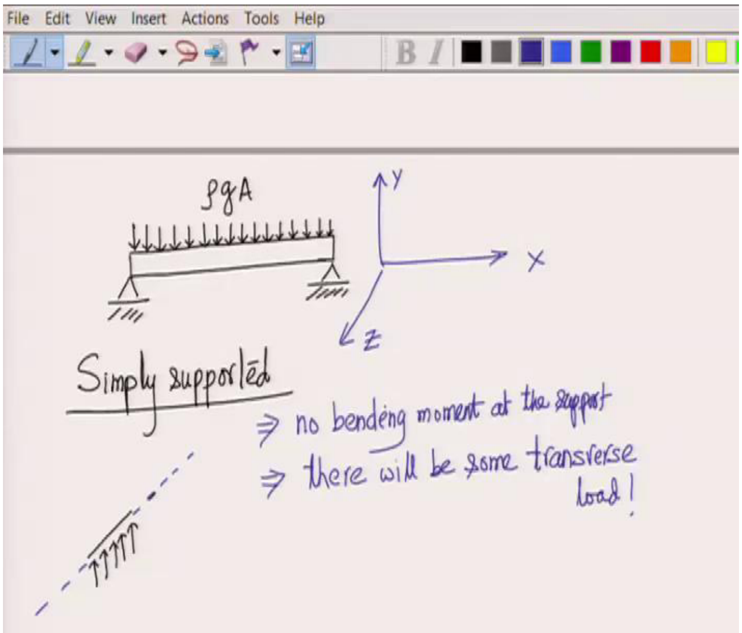

Figure 8: A constant distributed load is applied on a simply supported beam - a zoomed view of the line contact of beam with right support is also shown

Unlike clamped supports, simple supports do not exert reactive bending moment on the beam. They consist of line contacts. If we have a line contact between the beam and the support, the contact traction will obviously pass through the contact line as shown in Figure 8. So, the moment due to this line force about the contact line itself will always be zero.[2] Figure 8 shows a beam that is simply supported. As the line contacts at the support are along z axis, there is no reactive moment at the two ends about z axis. The beam is also subjected to a constant distributed load. To visualize this distributed load, we can also think of it as the weight of the beam itself in which case the distributed load will be equal to ρgA, where ρ and A denote the beam's density and cross-section area, respectively. Although, the reactive bending moment at the two ends is zero, the reactive transverse load will be present. This is because the beam is restricted by the support and cannot move downwards. This reactive transverse load will be equal to the integration of the traction over the line contact (see the zoomed view of line contact in Figure 8). As a rule of thumb, one can remember the following table while applying boundary condition for different kinds of support:

Transverse displacement	Transverse Load
0 ⇒	×
×	⇐ 0
Rotation(z)	**Moment(z)**
0 ⇒	×
×	⇐ 0

Here a × implies that the corresponding quantity is an unknown and may be non-zero. For example, if the applied transverse load at an end is zero, then the displacement at that end in the direction of the loading is an unknown. If the applied moment at an end is zero, then the rotation at that end is an unknown. For our beam (shown in Figure 8), the displacement of the beam in the y direction is restricted at the ends. So, the transverse load in the same direction is an unknown. Also, the bending moment about z axis is zero at the ends which implies the end-rotation about z axis is an unknown.[3] To obtain deflection, we have to solve equation (8) for which we again require the bending moment profile. So, we cut a section at a distance X from the left end of the beam and draw the free body diagram of the right part of the beam as shown in Figure 9.

[2] The clamped supports on the other hand have surface contacts: the traction is distributed over the entire contact surface and hence their resultant has a non-zero moment.

[3] In case of clamped supports, the rotation is restricted and hence the reactive moment is non-zero and an unknown.

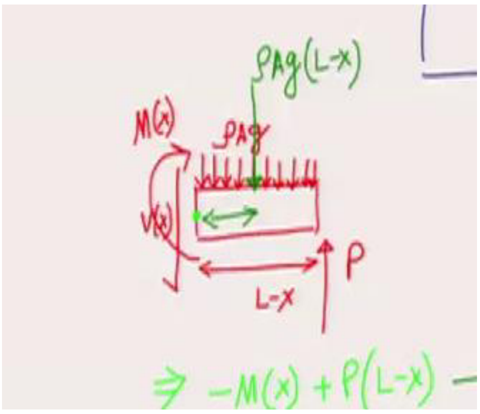

Figure 9: Free body diagram of the right part of the beam shown in Figure 8

On its left-end cross-section, shear force *V* (*X*) is directed downwards while the bending moment *M*(*X*) is clockwise. On the right-end cross-section, bending moment is zero while some unknown shear force *P* acts in the upward direction. In this problem, we have an extra unknown *P* for which we will require an extra boundary condition. Moment balance about the centroid of the left-end cross-section is now carried out. The contribution due to the distributed load is calculated by replacing the distributed load by a single equivalent force acting at the center of this part of the beam: the equivalent single force will be equal to magnitude of the distributed load times the length of the part. Thus, we finally get:

$$-M(X) + P(L-X) - \rho A g \frac{(L-X)^2}{2} = 0$$
$$\implies \quad M(X) = P(L-X) - \rho A g \frac{(L-X)^2}{2}. \tag{17}$$

We can now substitute the expression of *M*(*X*) in equation (8) to get

$$\frac{d^2 y}{dX^2} = \frac{P}{EI}(L-X) - \frac{\rho A g}{EI}\frac{(L-X)^2}{2} \tag{18}$$

As pointed earlier, we require one extra boundary condition (total 3) due to the presence of the unknown parameter *P*. The three boundary conditions are vanishing of transverse displacement at the two ends and vanishing of bending moment at the left end: we have already used vanishing of bending moment at right end in deriving (17). Upon setting *M*(0) = 0 in equation (17), we get

$$P = \frac{\rho A g L}{2} \tag{19}$$

This means that the reactive transverse load by the support is equal to half of the total load due to distributed force. Substituting the above expression of *P* in equation (18), we get

$$\frac{d^2 y}{dX^2} = \frac{\rho A g}{2EI}\left[L^2 - LX - L^2 - X^2 + 2LX\right] = \frac{\rho A g}{2EI}\left[LX - X^2\right] \tag{20}$$

Upon further integrating it twice, we obtain

$$y = \frac{\rho A g}{2EI}\left[\frac{LX^3}{6} - \frac{X^4}{12}\right] + C_1 X + C_2 \tag{21}$$

Finally, we use the following two remaining boundary conditions:

$$y(0) = 0, \qquad y(L) = 0 \tag{22}$$

to obtain (C_1,C_2) which yields

$$C_1 = -\frac{\rho A g L^3}{24EI}, \quad C_2 = 0 \tag{23}$$

2.3 Example 3 (start time: 48:36)

Let us now think of a beam which is clamped at both its ends. The beam sags down due its own weight as shown in Figure 10.

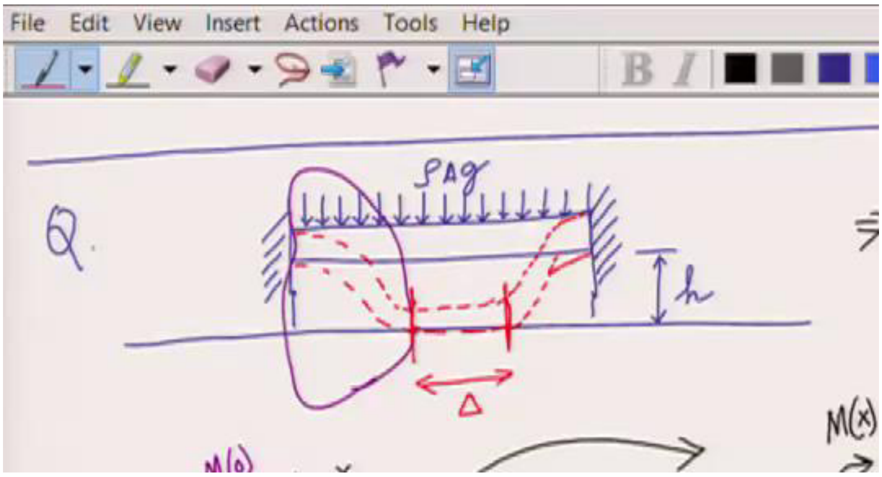

Figure 10: A beam clamped at its ends. It sags down due to its own weight but is restricted mid-way by the ground.

The distributed load is again *ρAg*. However, the ground position (*h* below the beam) is such that the some part of the beam rests on the ground upon deformation while the remaining part just hangs. We have to find the length of the beam Δ which will rest on the ground. First of all, we can notice that the problem is symmetrical with respect to the center of the beam ($X = \frac{L}{2}$). Let us consider the left hanging part of the beam, i.e., from $X = 0$ to $X = \frac{L-\Delta}{2}$. We draw its free body diagram as shown in Figure 11.

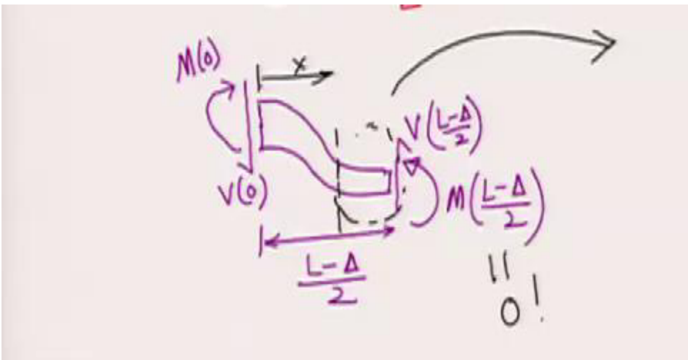

Figure 11: Free body diagram of the left hanging part of the beam shown in Figure 10.

Bending moment *M*(0) and shear force *V*(0) act on its left-end cross-section (applied by the clamped end). A shear force and a bending moment also acts on the right-end of this part (applied by the remaining part of the beam). All of these four quantities are also unknowns. To proceed with the beam equation (8), we need to first find the bending moment profile. We thus cut a section at a distance *X* from the left end and draw the free body diagram of the right part as shown in Figure 12.

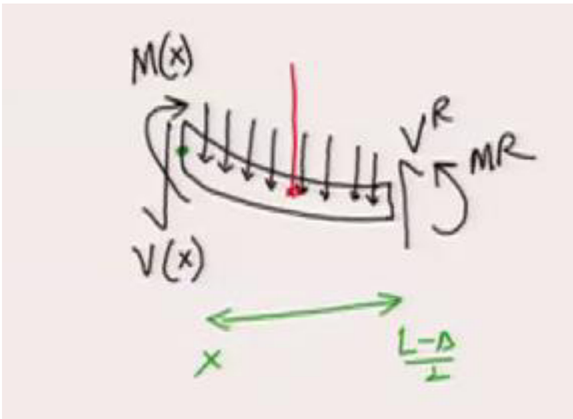

Figure 12: Free body diagram of a section of the hanging part shown in Figure 10 at a distance *X* from the left end.

Shear force and bending moment act on left and right-ends whereas distributed load acts throughout its length. Its force balance gives us

$$V(X) = V^R - \rho A g\left(\frac{L-\Delta}{2} - X\right) \tag{24}$$

We then do moment balance about the centroid of its left-end cross-section which yields

$$-M(X) + M^R + V^R\left(\frac{L-\Delta}{2} - X\right) - \frac{\rho A g}{2}\left(\frac{L-\Delta}{2} - X\right)^2 = 0$$
$$\implies \quad M(X) = M^R + V^R\left(\frac{L-\Delta}{2} - X\right) - \frac{\rho A g}{2}\left(\frac{L-\Delta}{2} - X\right)^2. \tag{25}$$

Plugging this in equation (8), we get

$$\frac{d^2y}{dX^2} = \frac{1}{EI}\left(M^R + V^R\left(\frac{L-\Delta}{2} - X\right) - \frac{\rho A g}{2}\left(\frac{L-\Delta}{2} - X\right)^2\right) \tag{26}$$

We now think of boundary conditions. As there are three additional unknown parameters (V^R, M^R, Δ), a total of five boundary conditions will be required. Out of these, two can be obtained as earlier from the clamped end at $X = 0$, i.e.,

$$y(0) = 0, \quad \frac{dy}{dX}(0) = 0 \tag{27}$$

We cannot use the clamped boundary condition of the original beam at $X = L$ because we are only analyzing the left hanging portion of the full beam. If we carefully observe, we can see that at $X = \frac{L-\Delta}{2}$, the beam starts to get into contact with the ground surface. Thus, the deflection of this point is $-h$, i.e.,

$$y\left(\frac{L-\Delta}{2}\right) = -h. \tag{28}$$

Also, as the ground is flat, the beam's deflection is uniform throughout the resting portion of the beam. Thus, the first and second derivatives of deflection in the resting portion will be zero which by continuity will also be zero at $X = \frac{L-\Delta}{2}$**.** Thus, we get our fourth boundary condition as

$$\frac{dy}{dX}\left(\frac{L-\Delta}{2}\right) = 0 \tag{29}$$

As the second derivative $\frac{d^2y}{dX^2}$ vanishes throughout in the resting region, the bending curvatue and hence the bending moment will be zero in the entire resting region of the beam. Thus, M^R which is the bending moment at the edge of the flat region must also be zero.[4] This gives us the final boundary condition, i.e.,

$$M^R = 0. \tag{30}$$

Using all the five boundary conditions in the EBT equation, we can solve for the deflection of the beam and also the length of the resting portion of the beam, i.e., Δ.

[4] There can also be no jump discontinuity in bending moment at $X = (L - \Delta)/2$ because of line contact of beam with ground due to which the ground does not exert any reactive bending moment at this point.