Design and Analysis of Algorithms, Chennai Mathematical Institute
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Week- 03
Module - 05
Lecture - 22
Applications of BFS and DFS

We have seen how to use Breadth First and Depth First search to explore whether there is a path from a source to a target vertex. But one can do a lot more with these two procedures.

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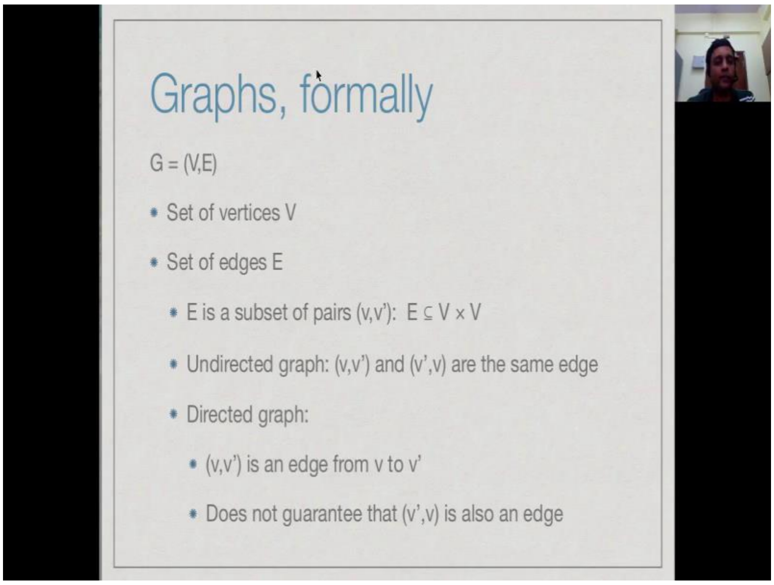

So, recall that a graph is a set of vertices and a set of edges which have connections between the vertices, and these maybe directed or undirected.

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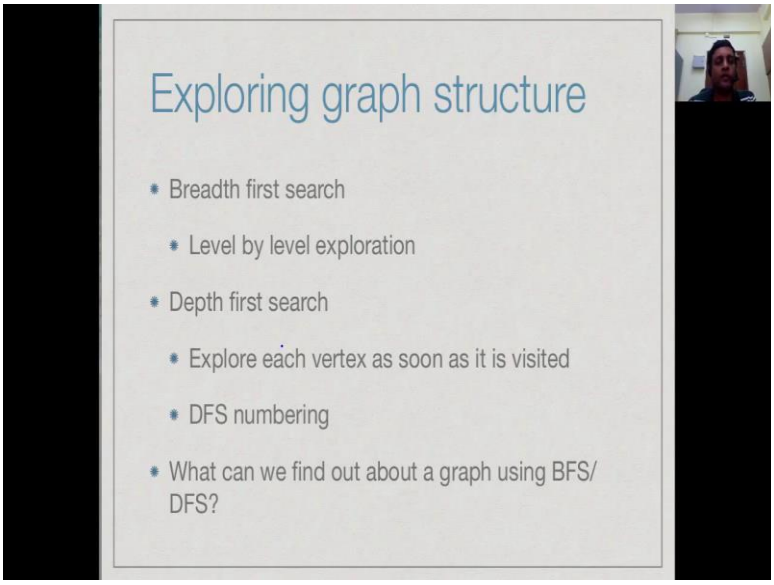

Now, when we do breadth first search, we do a level by level explorations starting at one vertex, when we do depth first search each time we go to a new vertex, we switch the exploration to that vertex and whenever we reach a dead end we backup. And one of the features of depth first searches that we can keep track of the order in which we enter in exit vertices in this recursive procedure. So, now let us see how we can use BFS and DFS to find out more about this structure of the underline graph.

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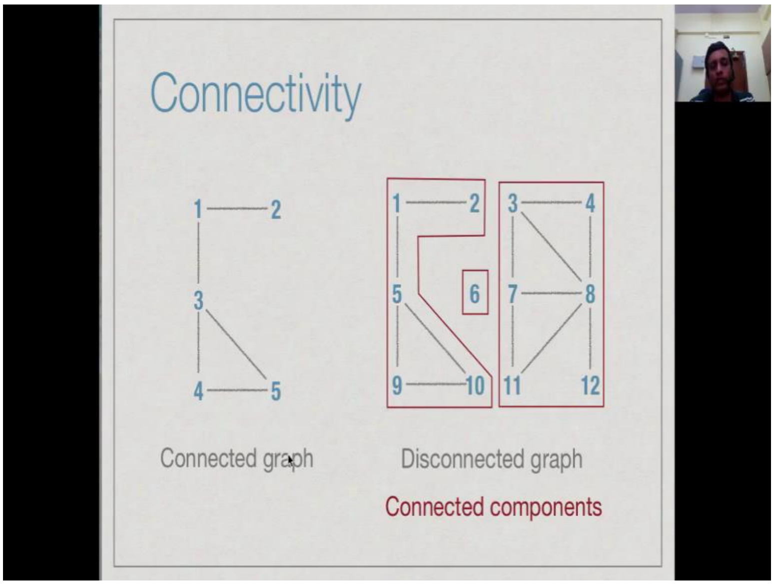

So, one fundamental property of undirected graph is whether or not disconnected, can we go from every vertex to every other vertex. So, you can see in these two pictures that the graph on the left is connected, because you can go from every vertex to every other vertex. On the other hand, on the right hand side some vertices cannot be reach from other vertices. For example, one cannot go from 2 to 7 or from 6 to anywhere else.

Now, when we have an undirected graph which is disconnected, we are also interested at finding out what the connected components are.... So, we want to grouped to gather those vertices which are connected to each other into one unit and find out which of these units are there and how many such units are there, in which vertices belong to the same unit.

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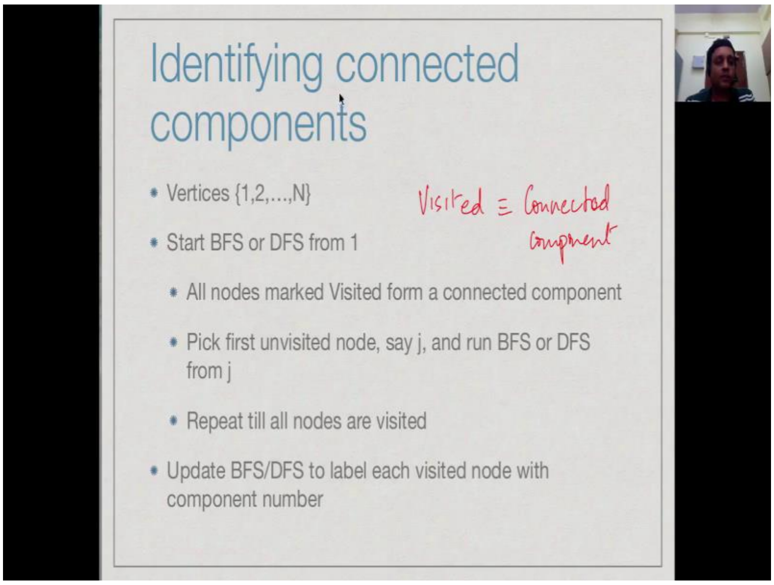

So, at first target is to used BFS or DFS to identify connected components. So, this is quite easy to do which start with the node label 1 or any other node. Now, we run BFS or DFS from this node and in this process we will mark a number of nodes as visited, at this point if there are any vertices which are not visited by BFS or DFS starting from the first node, this means that they do not belong to the same connected component. So, it is easy to show that what is mark visited is equal to the connected component.

So, the connected component containing the start node, so now we go back and we look at the first node in the list which is not mark visited and very start BFS or DFS from that node. So, we will get a new connected component, consisting of those vertices which are reachable from the first node which is marked and connected. Now, there are still nodes which are unvisited, we restart from one of those and go on. So, what we can do now is, we can label each DFS with a different numbers, so at the end we can associate with each vertex, the number of the component in which it we discovered.

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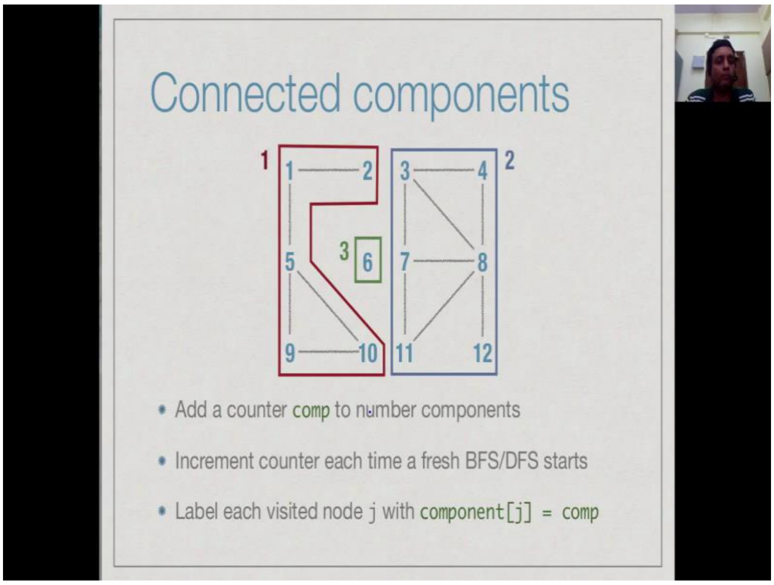

So, let us look at an example of this, so supposing we have this graph, we begin by having an extra variable in our BFS for example, or even DFS call compt to number the components. So, we initially said compt 1 and maybe we start are DFS or BFS from node 1. So, in this process we will visit 1, 2, 5, 9 and 10 and for all of these we will said component of j equal to comp. Now, this point we will realize that all nodes have been visited only 5 out of the 12 nodes have been visited.

So, we go to the smallest node which is not visited namely 3 and restart, but before we restart we update count to 2. Now, we start a BFS or a DFS said node 3 and visit everything that we can reach, and in this process we will identify these 6 nodes has being in component 2, at this point node 6 is still not mark visited. So, we restart a 3rd round of BFS or DFS with comp said to 3 that identify a 3rd component.

Now, there are no more unvisited nodes, so we can stopped and in there as a result of this repeated application of BFS and DFS, we have identify all the components and also clustered them. So, that all nodes in the same component or associated with the same component number.

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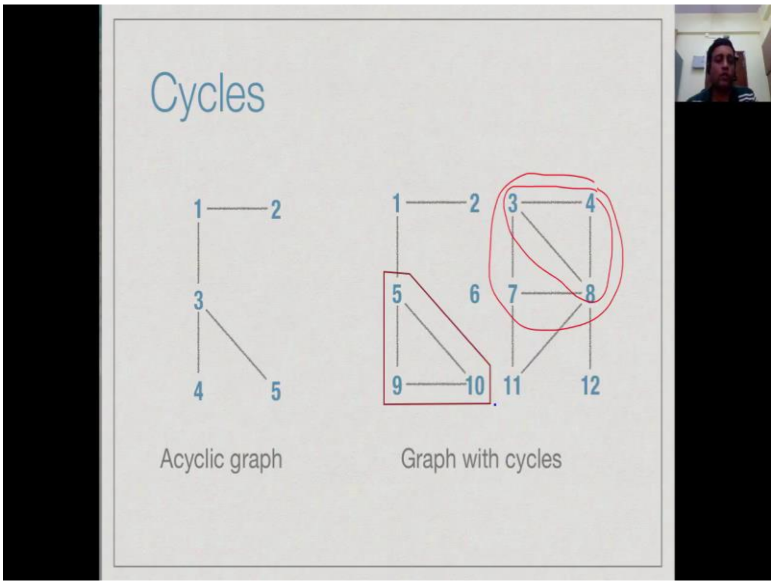

And other interesting structural property of a graph is whether or not it has cycles. So, a acyclic graph is a graph such as a left, in which you cannot started at any node and follow up sequence of edges and come back to the node and their right ((Refer time: 04:29)) graph with cycles. So, for instance 5, 9 and 10 forms cycles, there are also other cycles, there are several cycles for example, 3, 4, 7, 8 has a whole form a cycle with there are also smaller cycles with in unit click 3, 4, 8 and 3, 7, 8 and 7, 8, 11 is also a cycle and so on.

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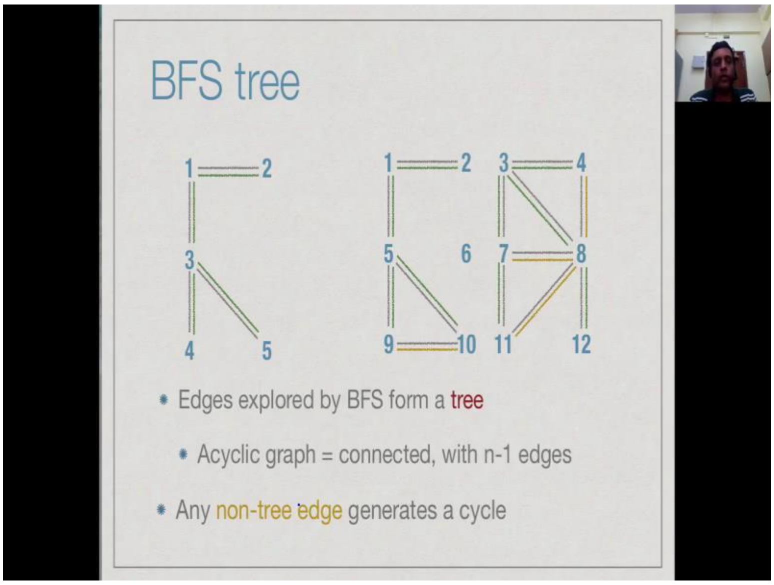

So, one of the things we can do when we execute BFS is to keep track of those edges which are actually used to mark vertices as visited. Now, if you are a acyclic graph such as one on the left, you can check that every node that is in the graph will actually be used as part of the BFS search. On the other hand, if you run BFS on a graph which has cycles then you will find that some edges are not use, because when you try to explore those edges, you find that target vertex is already visited.

For instance, since 10 is already visited as in neighbor of 5 when we start exploring 9 we do not use the edge 9, 10. Likewise, we do not use the edge 4, 8, because it is already visited as a set of neighbors of 3, remember in breadth first search we go to 3 in explore all it is 1 step neighbors. So, we will not directly 4, 8 and 7 has visited, so when we come to 4 we do not need to use the edge 4, 8 when we come to 7 we do not need to use 7, 8 and so on. So, there are these edges which are left out.

Now, it is easy to see that if we have a graph with n vertices and it is connected and it does not have cycles, then it will have exactly n minus 1 edges, this kind of a graph is called a tree. So, there are many definitions of trees, but a trees is basically a connected acyclic graph, connected means you can go from everywhere to everywhere, acyclic means there are no loops and any connected acyclic graph on n vertices will have exactly n minus 1 edges.

So, in any graph if we explore BFS, the edges that BFS actually uses will form a tree and this is called a BFS tree. Now, what happens about the remaining edges will these are called non tree edges, what you can check very easily is that any non tree edge will combine with the tree edges already there to form a cycle. In other words, when we run BFS if we find that there are some vertices or some edges rather which are not used, in other words there are any non tree edges then this graph will definitely have a cycle. So, having a cycle is to equivalent to finding a non tree edge while doing BFS, what is a non tree edge is just an edge will we come to explore i comma j and find the j is already be mark visited. So, we do not go to i, I do not use i comma j in BFS.

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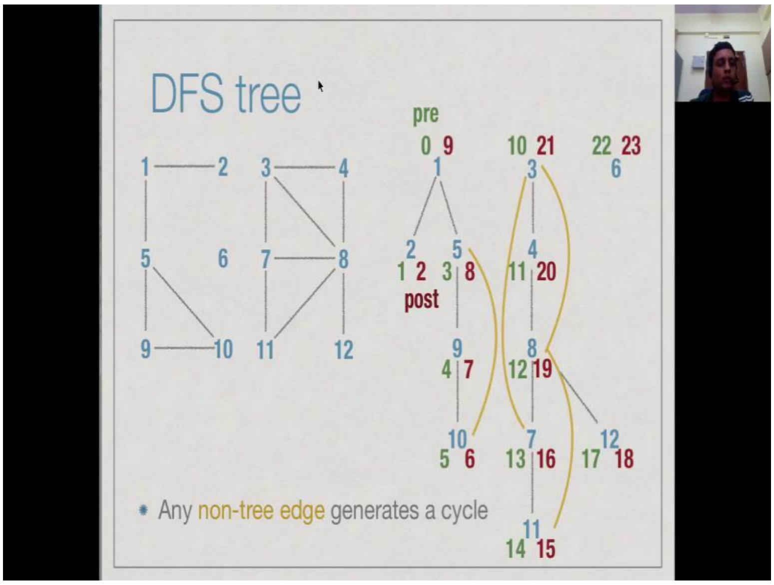

So, let us do the same thing, but DFS and let us compute the pre and post numbers. So, that we get some practice sets, so we sorted DFS at vertex 1 and we mark it is counter as 0. So, we enter vertex 0, vertex 1 at step 0, so this is the pre number of vertex 0. The first neighbor we explore from 1 is 2, so it has pre number 1, but 2 has no further successes. So, we exit from 2, so it has post number 2, so remember every time we enter we increment the counter, every time we exit we increment the counter.

So, we enter an increment exit vertex 2 in one step come back to 1 and explore it is next neighbor which is 5 which we enter at time 3, then we move vertex 9 at time 4 then from 9 we go to 10 at time 5, now 10 has no further neighbors to explore. So, we exit 10 at time 6, 9 has no further neighbors, so exit 9 at time 7 come back to 5, then 5 has no further neighbors, so we exit 5, and then finally we exit 1. So, at step 9 we have completed processing 1.

Now, we move to the first vertex which is not mark namely 3, we restart in new DFS from there. So, we enter 3 at time 10 from 3 will move to 4 at time 11, now 4 we move to 8 at time 12, from 8 we move to it smallest neighbors unvisited which is 7 at time 13 from 7 we go to 11 time 14, now 11 has no new neighbors to explore both 7 and 8 have been seen. So, we exit from 11, 7 has no more neighbors explore, so we exit from 8 visible have to explore 12. So, we enter 12 at step 17, then we exit from 12 and now 8 is finished, so we exit from 8.

Now, we come back to 4, 4; obviously, has no other vertices, so we come back to 3 and finally, we exit from 3 at time 21 at this 0.6 is still not marked. So, we start new DFS from 6, so we enter 6 at time 22 per 6 has no neighbors, so we exit at time 23. So, this like BFS generates a collection of trees. So, when we do DFS on a disconnected graph, each connected component will generated tree.

Now, if we look at the edges which we did not explore, these will again beet edges which are outside the tree. So, we can draw them on a different color, so we have the edge between 5 and 10 which we did not explore, because we explored 5 whether 10 directly from 9 and so on. So, once again just like an BFS, once we have finished DFS if there are non tree edges, then we have a cycle. So, both BFS and DFS on undirected graph can reveal a cycle through the presence of an non tree edge.

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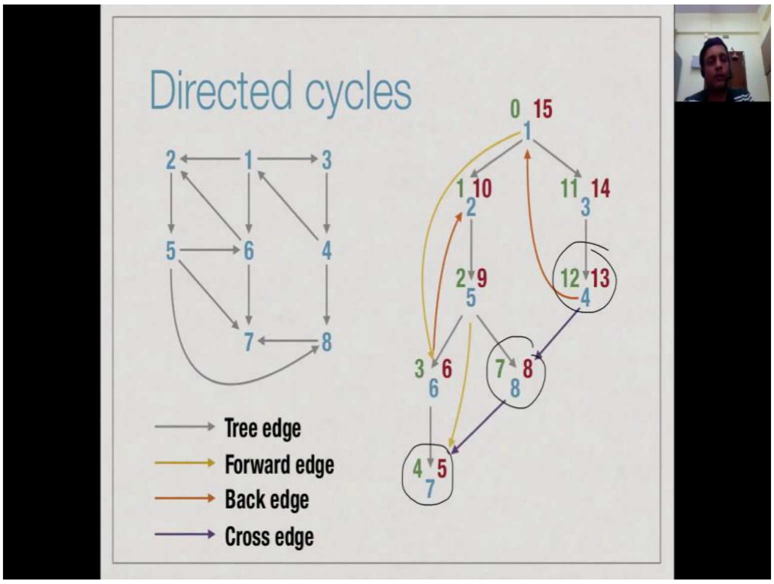

So, this situation with directed graphs is little more complement, so let us see what happens when we are cycles in directed graph. So, in a directed graph we need to follow the edge arrows, arrows along the edges. So, for example, 1, 3, 4, 1 which is cycle, because we can go around without changing direction, where as 1, 6, 2, 1 is not a cycle, because and the way back have to switch directions from 2 to 1 which I cannot go. So, let us do a DFS and see what this can tells up support cycles in this graph.

So, we begin with vertex 1 as usual, so 1 has free number 0 it is smallest neighbor is 2 in the smallest neighbor of 2 is 5, in the smallest neighbor of 5 is 6, in the smallest neighbor of 6 is 7, now from 7 that are no outgoing edges. So, we back track to 6 from 6 the only node that we can go to is 2 which have seen before, so we leaves 6. Now, we come to 5, 5 still has in outgoing edge which is 8, so we come to a 8.

Now, from 8 we cannot do anything, so we return from 8 back to 5, now 5 has nothing left to explores. So, we leave 5 likewise we leave 2 finally, we come back to 1, now at 1 we explore this left path. So, now we can we do not look were we look and other reduction go to 3. So, we explore 3, 3 will explore 4, but 4 cannot go to 8 or 1, because 1 has already been seen and source 8, so 4 will exit, so 3 will exit and then 1 will exit.

So, this happens to be a single connected graph, but it has cycles, so now if first look at these edges, the edges at drawn are tree edges as before. Now, if you look at the edges that the not been part of the graph, they fall into 3 groups. So, the first type of edge which is not the part of the tree is what we call a forward edge, so a forward edge is an edge which goes form a node to a node below it in the tree. So, we have a node from 1 to 6 for example,. So, this edge is a tree edge is not a tree edge, but it is a forward edge because 1 was about 6 in the part.

Likewise the node from 5 to 7, because we actually explore this graph is 5, 6, 7, so 5 to 7 is not tree. So, these are forward edges, the other category of edges which are there in the graph which are not in the tree have backward edges, they go up the tree. So, from 6 there is the edge back to 2 which we did not use, because two at already been explore. Likewise from 4 back to 1 there is edge we did not explore, because that was already there.

There is another category of edges which are not there in the tree, but which there in the graph and these are edges such as from 6 back to 2. So, this is an edge from a later vertex to a earlier vertex call from 4 back to 1, so these are what are called back edges. So, back edge in the graph which in the DFS tree goes from a lower vertex to a higher vertex. And finally, there are some edges which are neither going forward no backward, but sideways. So, these are edges like 8 to 7 and from 4 to 8, so this cross, so 4 is not below 8 nor is 8 below 4, 7 is not below 8 because there both below 5 and so on.

So, these we call cross edges, now it is easy to argue that a cross edge will only go from right to left. In other words, it only go from higher number to lower number, because if you wanted draw an edge like this for instance, then this would mean, but there was an edge from 2 to 4. So, we would explore 4th root to rather than weight go back one and explore. So, we cannot have grass edges which go from lower numbers to higher number, it must go from higher numbers to lower numbers.

So, now we have not one, but 3 types of non tree edges, unlike the directed case when we had the clear distinction between 3 edges and non tree edges, here we have 3 types of non tree edges. Now, which once of these correspond to cycles, so if I look at this 1, 3 edge, so this 1 6 edge. So, this 1 6 edge does not actually create a cycle, because we already saw the this is not a cycle.

So, in order for it to complete the cycle, it must be with case that there is a path including the set which found a directed cycle. Now, it is easy to see that the only situation where this will actually happen with there is back edge, for there is a back edge we know that there is a path coming from 2 down to 6 and then their following the back edge this forms a directed cycle.

Likewise, we know that there is a path coming from 1 down to 4 in the following back edge we follow directly some. On the other hand, if we look at the other types of edges for it is means, then we have this path here for this is parallel to the other path here. So, together these are both two different ways are going to 1 to 6, but there not a hide, in same way we can see that if we have a cross edge like this, then we have some path coming from here and some path going from there.

But, again these two different ways at reaching 7 from 5 and they are not relies a cycle. So, in terms of by little analysis that only back edges form cycles and this is actually something that you can prove we will not to prove formally, but it is on argued to way we did just now.

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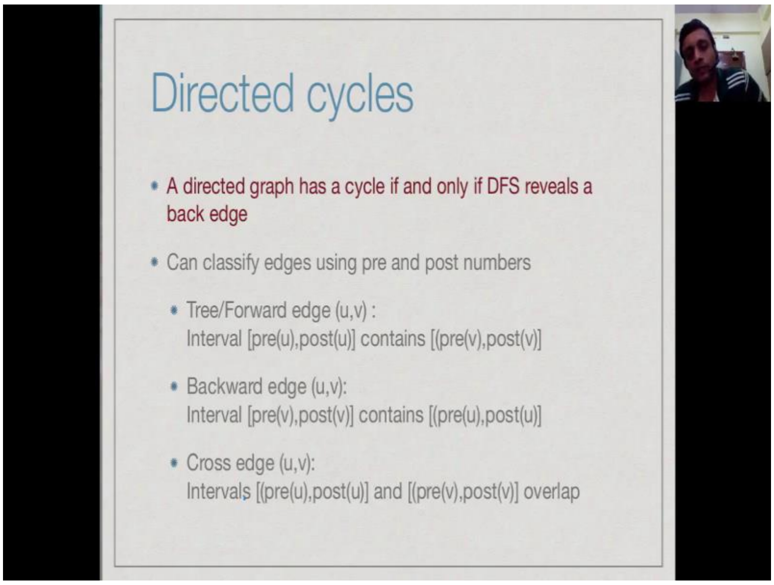

But, a directed graph has a cycle if and only DFS will prevails of back edge, now it terms out that these pre and post numbering are very useful to help as classify with types of edges that are there in the graph. So, for both tree and forward edges, so you will notice that if you go back to this numbering ((Refer Time: 16:19)) that these things form a integral that you can think of this as from 0 to 15, from set 0 to 15 as which exploring them, from step 11 to 14 have it is exploring 3 from step 2 to 9, I would exploring 5 and so on.

So, if you look at the pre and the post number which has I started exploring the number at the pre and I finished exploring at post and everything else that was below happen in between. So, for a forward edge we internal about will be bigger than the interval below. Because, I start a went below during this period I came back before it ended. So, for inserts the forward edge from 0, 15 to 3 comma 6. So, we integral 3, 6 is inside 0, 15 this is also ((Refer Time: 17:05)) true for tree edges, because in the tree edge also I am going the forwards I enter then lower node after I enter this. So, it is starting point to be later and it is ending point to clearly.

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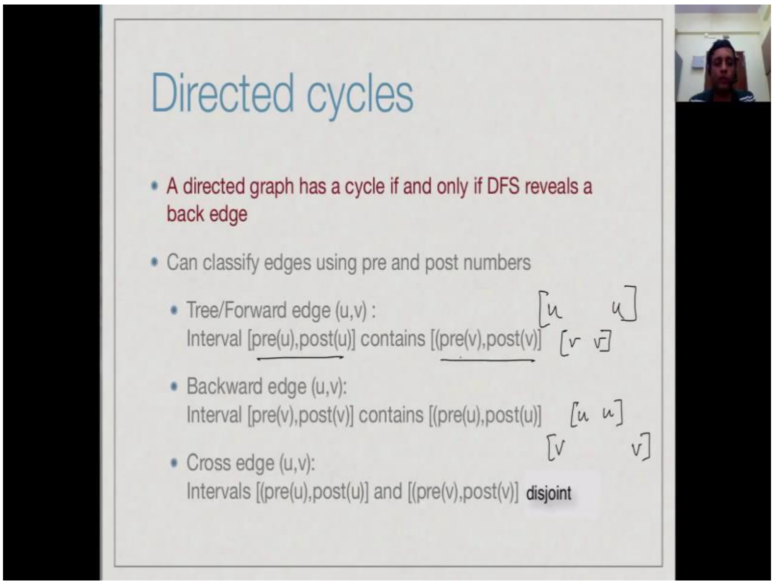

So, for both tree edges and forward edges, if I am going from u to v then the interval with the start node u will contain the other one. But, this will be sitting inside this one, pre v post v will be sitting inside pre u post u, so I will have this picture. So, this will be internal pre for you and this is interval pre. Conversely is exactly the opposite for backward edges a start comma smaller interval and at go to bigger interval. So, the smaller integral will be the starting at point with edge, then the bigger integral will be the ending point.

So, if I look at an edge in my DFS tree and if I look at the tree and post numbers are associated with the end point I can determine words of forward edge and backward edge and finally, it will turn out that for cross edges in the intervals are disjoint ((Refer Time: 18:02)). So, we can see here that we are finished processing 4 vertex 7 before we get to 8. So, there is no intersection between the interval 7, 8 and 4, 5 likewise we are finished processing 8 before we went to 4 that is why it is a cross edge, they are on different branches of the tree.

So, there is no intersection 7, 8 and 12, 13, so therefore a directed graph has a cycle if only if DFS reveals a back edge, and we can classify edges has being forward edges, backward edges or cross edges by just looking at the pre and post numbering of the end points of the edge.

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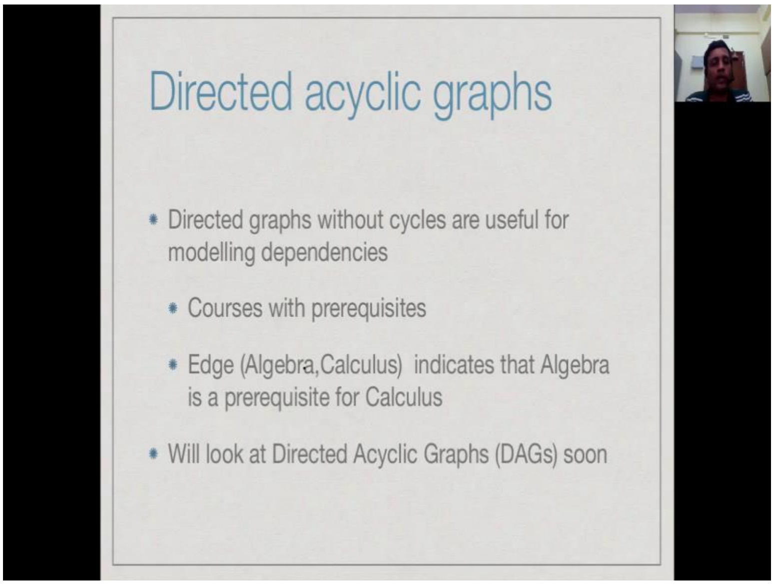

Now, it is important to identify cycles, because we do not have cycle we have a very nice class of graphs called directed a cyclic graphs is a useful for modeling dependencies. For instance if you want to list out a bunch of courses which are being offered and they have pre requisites, then and natural way to model this is using at directed graph with the edges represent pre requisites, for instance if they have an edge from algebra to calculus with indicates algebra is a pre requisite for calculus, it will not have cycles.

Because we cannot have two courses which are pre requisites for each other; otherwise, we will not we did input take either inputs. So, we will look a directed acyclic graphs or DAGs soon in a later lecture.

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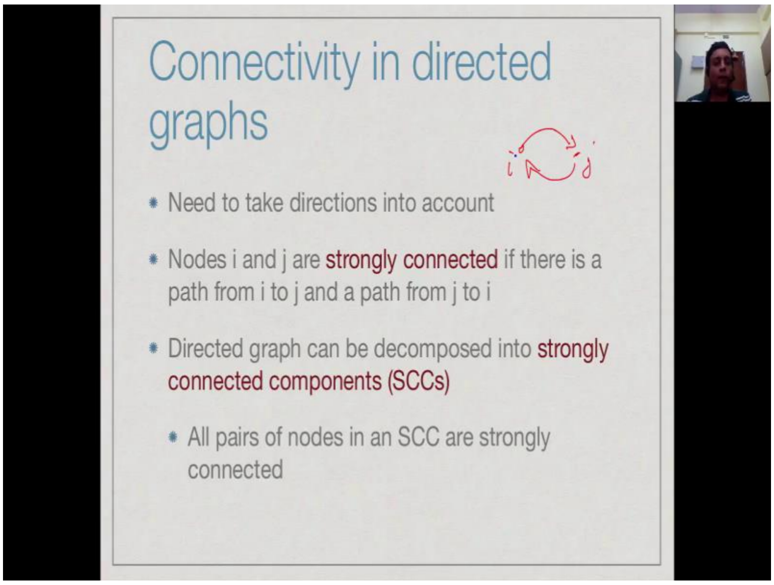

What about connectivity in directed graphs? So connectivity in a directed graph is not just a question at having these edges between the graphs, but having them in the right direction. So, we say that two nodes are strongly connected, if I can go from i to j by a path and I can come back from j to i ((Refer Time: 19:38)). So, is not enough to just have edges in some has a direction I must be able to go from i to j and come back from j to i in which case has a strongly connected.

So, it turns out that the directed graph can always be decompose in what are calls strongly connected components. A strongly connected graph has a property that every pair of nodes in that component is strongly connected from every node in the component you can go to every other node in the component and come back.

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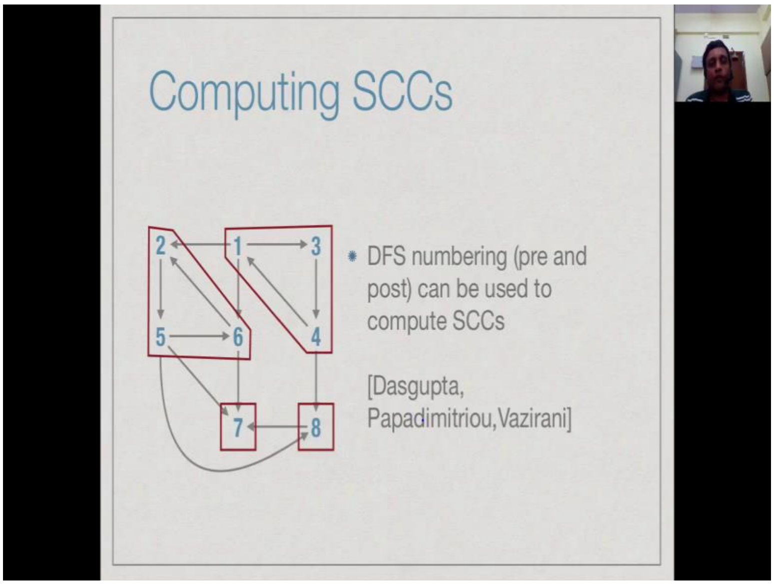

So, for instance it will look at this graph then strongly connected components, one is the cycle 1, 3, 4 we can go from 1, 2, 3 to 4 and come back. So, from ((Refer Time: 20:14)) this cycle we can come back to any other node. Likewise 2, 5 and 6 forms are strongly connected component, 7 on it shown a strongly connected component. Because, we cannot go anywhere and 8 also is strongly connected component, because we leave yet we cannot come back to it way this graphic structure.

So, this graph as 4 strongly connected components, have it turns out the DFS numbering using pre and post numbers can be used compute strongly connect components a very elegant algorithm is given and the book by Dasgupta property Papadimitriou and Vazirani and if you are interested you can look it up in that book.

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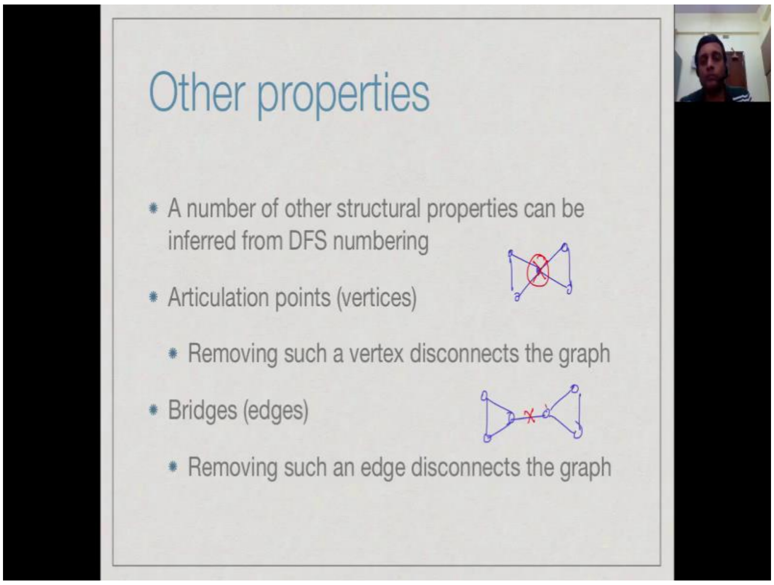

So, we have seen some concrete examples of what you can compute, there are many other properties that you can compute using BFS and DFS. For instance, there are these things called articulation points, if you are graph looks like this where I have some vertex which is a crucial vertex, if I remove this vertex this graph also upon to disconnect components, I can identify such vertices using BFS and DFS and particular using DFS.

Similarly, if I have the situation where I have an edge like this, where if I remove this edge then the graph gets disconnected, then I can again identify such an edge using DFS. Now, these are important, because if these represents some kind of communication network or some root network, in these are bottle necks, these are critical points, if this is an intersection and there is an accident no traffic and go from many part on left and many part on right or this is a network wire this cable gets cut then the network will get cut disconnected due to components.

So, these kinds of properties can also be computed during BFS and DFS. So, it is important to realize therefore, where BFS and DFS is not just for connecting to finding out whether vertex s can reach vertex t, you can get a well information and these are linear time algorithms and these are all operations which can be perform during BFS and DFS. So, very efficiently you can compute various properties in the graph and use these two exploit these will design more efficient procedures or to identify other things that need to be done.