Chapter 29: Eigenvalues

Introduction

In civil engineering, many problems related to structural stability, vibration analysis, and systems of differential equations involve matrices. Among the most powerful tools for analyzing matrices are **eigenvalues** and **eigenvectors**. These concepts help us understand linear transformations, especially those that involve stretching, compressing, or rotating vectors. The knowledge of eigenvalues is essential for applications such as stability analysis of structures, principal stress analysis, and modal analysis in vibration problems.

This chapter explores the theoretical foundations and computational techniques of eigenvalues, especially as they apply to real symmetric matrices, which often arise in civil engineering contexts.

29.1 Definitions and Concepts

29.1.1 Eigenvalues and Eigenvectors

Let A be an $n \times n$ square matrix. A scalar λ is called an **eigenvalue** of A if there exists a non-zero vector $\mathbf{x} \in \mathbb{R}^n$ such that:

$$A\mathbf{x} = \lambda\mathbf{x}$$

Here:

- $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) is an **eigenvalue**,
- $\mathbf{x}$ is a **corresponding eigenvector**.

This equation can be rearranged as:

$$(A - \lambda I)\mathbf{x} = 0$$

This is a **homogeneous system** of linear equations. For a non-trivial solution ($\mathbf{x} \neq 0$), the coefficient matrix must be **singular**, i.e.,

$$\det(A - \lambda I) = 0$$

This is called the **characteristic equation**, and the polynomial $\det(A - \lambda I)$ is the **characteristic polynomial**.

29.2 Computing Eigenvalues and Eigenvectors

29.2.1 Steps to Find Eigenvalues

1. Start with the square matrix A.
2. Subtract λI from A to get $A - \lambda I$.
3. Compute the determinant $\det(A - \lambda I)$.
4. Solve the resulting characteristic polynomial $p(\lambda) = 0$ for λ. These are the eigenvalues.

29.2.2 Steps to Find Eigenvectors

For each eigenvalue λ:

1. Substitute λ into $A - \lambda I$.
2. Solve $(A - \lambda I)\mathbf{x} = 0$ to find the null space (eigenvectors).

29.3 Algebraic and Geometric Multiplicities

Algebraic Multiplicity

The **algebraic multiplicity** of an eigenvalue λ is the number of times λ appears as a root of the characteristic polynomial.

Geometric Multiplicity

The **geometric multiplicity** is the dimension of the eigenspace corresponding to λ, i.e., the number of linearly independent eigenvectors associated with λ.

Important:

$$1 \leq \text{Geometric Multiplicity} \leq \text{Algebraic Multiplicity}$$

29.4 Properties of Eigenvalues

1. **Trace and Determinant:**
    - The **sum** of eigenvalues (counted with multiplicity) is equal to the **trace** of the matrix:

$$\sum \lambda_i = \text{tr}(A)$$

- The **product** of eigenvalues is equal to the **determinant**:

$$\prod \lambda_i = \det(A)$$

2. **Eigenvalues of Triangular Matrices:**
   - For an upper or lower triangular matrix, the eigenvalues are simply the diagonal entries.
3. **Real Symmetric Matrices:**
   - Have **real** eigenvalues.
   - Are **orthogonally diagonalizable** (important in principal stress/strain analysis).

29.5 Diagonalization of a Matrix

A matrix A is **diagonalizable** if there exists an invertible matrix P and a diagonal matrix D such that:

$$A = PDP^{-1}$$

Where:

- The columns of P are eigenvectors of A,
- The diagonal entries of D are the corresponding eigenvalues.

Diagonalization is useful in:

- Solving systems of differential equations,
- Matrix powers: $A^k = PD^kP^{-1}$,
- Vibration analysis in civil engineering.

29.6 Applications in Civil Engineering

1. Structural Analysis

In stiffness matrix methods, eigenvalues represent natural frequencies of structures. Large eigenvalues indicate stiff modes, while small ones indicate flexible modes.

2. Stability of Structures

Buckling analysis involves eigenvalue problems:

$$(K - \lambda G)\mathbf{x} = 0$$

Where K is the stiffness matrix and G is the geometric stiffness matrix.

3. Modal Analysis

The dynamic behavior of buildings, bridges, and other structures is analyzed using eigenvalue problems. The eigenvectors represent **mode shapes** and the eigenvalues represent **natural frequencies**.

4. Principal Stresses and Strains

In stress analysis, the **stress tensor** is symmetric. The eigenvalues of the stress matrix are the **principal stresses**, and the eigenvectors indicate the principal directions.

29.7 Special Case: Symmetric Matrices

Let $A \in \mathbb{R}^{n \times n}$ be symmetric ($A = A^T$). Then:

- All eigenvalues are **real**.
- Eigenvectors corresponding to **distinct eigenvalues are orthogonal**.
- A can be diagonalized by an **orthogonal matrix**:

$$A = QDQ^T$$

This is especially relevant in **stress analysis** and **finite element methods (FEM)**.

29.8 Example Problems

Example 1: Eigenvalues and Eigenvectors

Let:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Step 1: Find characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = 0$$

$$\Rightarrow \lambda = 1, 3$$

Step 2: Find eigenvectors:

- For $\lambda = 1$: $(A - I)\mathbf{x} = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- For $\lambda = 3$: $(A - 3I)\mathbf{x} = 0 \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

29.9 Numerical Methods (Brief Introduction)

For large matrices arising from FEM or real-world structural models, numerical techniques are used:

- **Power Method**: Estimates the largest eigenvalue.
- **QR Algorithm**: Computes all eigenvalues numerically.
- **Jacobi Method**: For symmetric matrices.

These are implemented in civil engineering software tools like ANSYS, STAAD.Pro, and MATLAB.

29.10 Power Method for Dominant Eigenvalue

The **Power Method** is a numerical iterative algorithm for finding the **dominant eigenvalue** (i.e., the eigenvalue of largest magnitude) and its corresponding eigenvector of a matrix. This is especially useful when the matrix is large and sparse, which is often the case in finite element models of civil structures.

Algorithm Steps:

Given a matrix $A \in \mathbb{R}^{n \times n}$, and an initial guess vector $\mathbf{x}_0$, perform the following:

1. Normalize $\mathbf{x}_0$, i.e., $\mathbf{x}_0 := \frac{\mathbf{x}_0}{\|\mathbf{x}_0\|}$
2. For $k = 1, 2, \ldots$:
   - $\mathbf{y}_k = A\mathbf{x}_{k-1}$
   - $\mu_k = \| \mathbf{y}_k \|$
   - $\mathbf{x}_k = \frac{\mathbf{y}_k}{\|\mathbf{y}_k\|}$

As $k \to \infty$, $\mathbf{x}_k \to \mathbf{x}$ (an eigenvector), and $\mu_k \to \lambda$ (the dominant eigenvalue).

Convergence Conditions:

- The matrix must have a unique eigenvalue λ_1 such that $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$.
- The initial vector $\mathbf{x}_0$ must have a component in the direction of the dominant eigenvector.

Use in Civil Engineering:

Used for estimating the **fundamental natural frequency** (first mode shape) of tall buildings and long bridges modeled by stiffness and mass matrices.

29.11 QR Algorithm for Eigenvalue Computation

The **QR algorithm** is a robust numerical method to compute **all** eigenvalues (and optionally eigenvectors) of a square matrix.

Basic Idea:

If $A_0 = A$, then:

$$A_k = Q_k R_k, \quad A_{k+1} = R_k Q_k$$

Where:

- Q_k is orthogonal, R_k is upper triangular (via QR decomposition),
- This process iterates: $A_{k+1} = Q_k^T A_k Q_k$.

As $k \to \infty$, A_k converges to an upper triangular matrix with eigenvalues of A on the diagonal.

Advantages:

- Applicable to general square matrices.
- Works well with symmetric matrices (faster convergence).

Civil Engineering Use:

Used in vibration analysis software to compute full mode shapes of complex structural systems.

29.12 Cayley-Hamilton Theorem

The **Cayley-Hamilton Theorem** states that every square matrix satisfies its own characteristic equation.

Let $A \in \mathbb{R}^{n \times n}$ and its characteristic polynomial be:

$$p(\lambda) = \det(A - \lambda I) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda + a_n$$

Then:

$$p(A) = A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I = 0$$

Applications:

- Matrix functions and inversion.
- Efficient computation of matrix powers.
- Reduction of high-order differential systems.

Civil Engineering Use: Simplifies calculations involving repeated matrix multiplications in dynamic simulations of structures.

29.13 Spectral Decomposition (For Symmetric Matrices)

If $A \in \mathbb{R}^{n \times n}$ is symmetric, then:

$$A = Q \Lambda Q^T$$

Where:

- Q is an orthogonal matrix whose columns are normalized eigenvectors,
- Λ is a diagonal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$.

Spectral Theorem (Real Symmetric Case):

Every real symmetric matrix is diagonalizable by an orthogonal transformation.

This is fundamental in **principal component analysis (PCA)** and **stress/strain tensor decomposition**.

29.14 Application: Principal Stress and Strain in 2D

In 2D solid mechanics, the **stress tensor** is given by:

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

To find **principal stresses**, solve:

$$\det(\sigma - \lambda I) = 0 \Rightarrow \lambda^2 - (\sigma_x + \sigma_y)\lambda + (\sigma_x \sigma_y - \tau_{xy}^2) = 0$$

Eigenvalues λ_1, λ_2 are the **principal stresses**.

The directions (eigenvectors) show the orientation of the **principal planes**, which are important in:

- Reinforcement design,
- Earthquake stress analysis,
- Tunnel lining stability.

29.15 Generalization to Complex Matrices and Systems

In some advanced civil engineering applications (e.g., damped vibrations, electrical network modeling), matrices may have **complex entries**. The theory of eigenvalues extends naturally to complex matrices:

- Eigenvalues can be complex.
- Complex eigenvectors arise in oscillatory and damping solutions.

Important tools:

- **Hermitian matrices** (analog of symmetric matrices over complex numbers).
- **Unitary diagonalization**: $A = U\Lambda U^*$

29.16 Eigenvalue Condition Number and Sensitivity

The **condition number** of an eigenvalue problem measures how sensitive eigenvalues are to changes in the matrix.

Definition:

If A is perturbed slightly to $A + \Delta A$, the eigenvalues λ_i may shift significantly, especially for non-symmetric matrices.

Civil Engineering Relevance:

In numerical simulations (like FEM), round-off errors and mesh imperfections can lead to eigenvalue drift:

- Can affect modal frequencies and buckling loads.
- Makes it essential to use well-conditioned models and high-precision computations.
