Chapter 13

DIRECT STIFFNESS METHOD

13.1 Introduction

13.1.1 Structural Idealization

1 Prior to analysis, a structure must be idealized for a suitable mathematical representation. Since it is practically impossible (and most often unnecessary) to model every single detail, assumptions must be made. Hence, structural idealization is as much an art as a science. *Some* of the questions confronting the analyst include:

1. Two dimensional versus three dimensional; Should we model a single bay of a building, or the entire structure?
2. Frame or truss, can we neglect flexural stiffness?
3. Rigid or semi-rigid connections (most important in steel structures)
4. Rigid supports or elastic foundations (are the foundations over solid rock, or over clay which may consolidate over time)
5. Include or not secondary members (such as diagonal braces in a three dimensional analysis).
6. Include or not axial deformation (can we neglect the axial stiffness of a beam in a building?)
7. Cross sectional properties (what is the moment of inertia of a reinforced concrete beam?)
8. Neglect or not haunches (those are usually present in zones of high negative moments)
9. Linear or nonlinear analysis (linear analysis can not predict the peak or failure load, and will underestimate the deformations).
10. Small or large deformations (In the analysis of a high rise building subjected to wind load, the moments should be amplified by the product of the axial load times the lateral deformation, $P - \Delta$ effects).
11. Time dependent effects (such as creep, which is extremely important in prestressed concrete, or cable stayed concrete bridges).

(such as beam element), while element 2 has a code 2 (such as a truss element). Material group 1 would have different elastic/geometric properties than material group 2.

Group No.	Element Type	Material Group
1	1	1
2	2	1
3	1	2

Table 13.3: Example of Group Number

6 From the analysis, we first obtain the nodal displacements, and then the element internal forces. Those internal forces vary according to the element type. For a two dimensional frame, those are the axial and shear forces, and moment at each node.

7 Hence, the need to define two coordinate systems (one for the entire structure, and one for each element), and a sign convention become apparent.

13.1.3 Coordinate Systems

8 We should differentiate between 2 coordinate systems:

Global: to describe the structure nodal coordinates. This system can be arbitrarily selected provided it is a Right Hand Side (RHS) one, and we will associate with it upper case axis labels, X, Y, Z, Fig. 13.1 or 1,2,3 (running indeces within a computer program).

Local: system is associated with each element and is used to describe the element internal forces. We will associate with it lower case axis labels, x, y, z (or 1,2,3), Fig. 13.2.

9 The x-axis is assumed to be along the member, and the direction is chosen such that it points from the 1st node to the 2nd node, Fig. 13.2.

10 Two dimensional structures will be defined in the X-Y plane.

13.1.4 Sign Convention

11 The sign convention in structural analysis is completely different than the one previously adopted in structural analysis/design, Fig. 13.3 (where we focused mostly on flexure and defined a positive moment as one causing “tension below”. This would be awkward to program!).

12 In matrix structural analysis the sign convention adopted is consistent with the prevailing coordinate system. Hence, we define a positive moment as one which is counter-clockwise, Fig. 13.3

13 Fig. 13.4 illustrates the sign convention associated with each type of element.

14 Fig. 13.4 also shows the geometric (upper left) and elastic material (upper right) properties associated with each type of element.

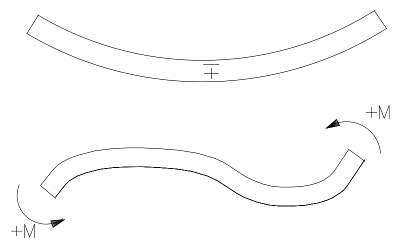

Figure 13.3: Sign Convention, Design and Analysis

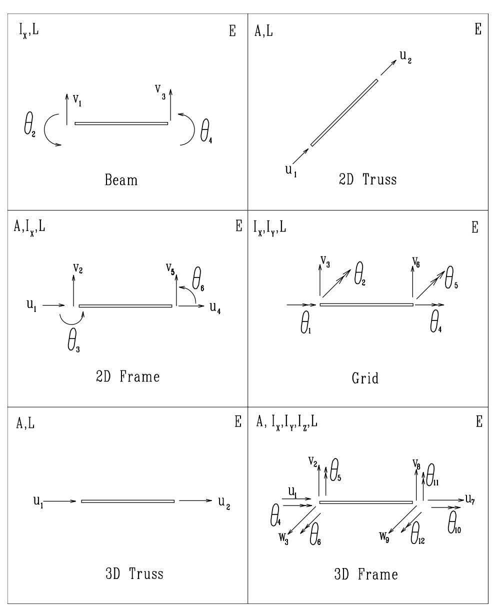

Figure 13.4: Total Degrees of Freedom for various Type of Elements

Type		Node 1	Node 2	$[\mathbf{k}]$	$[\mathbf{K}]$
				(Local)	(Global)
1 Dimensional					
Beam	$\{\mathbf{p}\}$	F_{y1}, M_{z2}	F_{y3}, M_{z4}	4×4	4×4
	$\{\boldsymbol{\delta}\}$	v_1, θ_2	v_3, θ_4		
2 Dimensional					
Truss	$\{\mathbf{p}\}$	F_{x1}	F_{x2}	2×2	4×4
	$\{\boldsymbol{\delta}\}$	u_1	u_2		
Frame	$\{\mathbf{p}\}$	F_{x1}, F_{y2}, M_{z3}	F_{x4}, F_{y5}, M_{z6}	6×6	6×6
	$\{\boldsymbol{\delta}\}$	u_1, v_2, θ_3	u_4, v_5, θ_6		
Grid	$\{\mathbf{p}\}$	T_{x1}, F_{y2}, M_{z3}	T_{x4}, F_{y5}, M_{z6}	6×6	6×6
	$\{\boldsymbol{\delta}\}$	θ_1, v_2, θ_3	θ_4, v_5, θ_6		
3 Dimensional					
Truss	$\{\mathbf{p}\}$	F_{x1},	F_{x2}	2×2	6×6
	$\{\boldsymbol{\delta}\}$	u_1,	u_2		
Frame	$\{\mathbf{p}\}$	F_{x1}, F_{y2}, F_{y3}, T_{x4} M_{y5}, M_{z6}	F_{x7}, F_{y8}, F_{y9}, T_{x10} M_{y11}, M_{z12}	12×12	12×12
	$\{\boldsymbol{\delta}\}$	u_1, v_2, w_3, θ_4, θ_5 θ_6	u_7, v_8, w_9, θ_{10}, θ_{11} θ_{12}		

Table 13.4: Degrees of Freedom of Different Structure Types Systems

13.2 Stiffness Matrices

13.2.1 Truss Element

22 From strength of materials, the force/displacement relation in axial members is

$$\begin{aligned} \sigma &= E\epsilon \\ \underbrace{A\sigma}_{P} &= \frac{AE}{L}\underbrace{\Delta}_{1} \end{aligned} \tag{13.1}$$

Hence, for a unit displacement, the applied force should be equal to $\frac{AE}{L}$. From statics, the force at the other end must be equal and opposite.

23 The truss element (whether in 2D or 3D) has only one degree of freedom associated with each node. Hence, from Eq. 13.1, we have

$$\boxed{[\mathbf{k}^t] = \frac{AE}{L} \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] \end{array}} \tag{13.2}$$

13.2.2 Beam Element

24 Using Equations 12.10, 12.10, 12.12 and 12.12 we can determine the forces associated with each unit displacement.

$$[\mathbf{k}^b] = \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \left[\begin{array}{cccc} \text{Eq. } 12.12(v_1 = 1) & \text{Eq. } 12.12(\theta_1 = 1) & \text{Eq. } 12.12(v_2 = 1) & \text{Eq. } 12.12(\theta_2 = 1) \\ \text{Eq. } 12.10(v_1 = 1) & \text{Eq. } 12.10(\theta_1 = 1) & \text{Eq. } 12.10(v_2 = 1) & \text{Eq. } 12.10(\theta_2 = 1) \\ \text{Eq. } 12.12(v_1 = 1) & \text{Eq. } 12.12(\theta_1 = 1) & \text{Eq. } 12.12(v_2 = 1) & \text{Eq. } 12.12(\theta_2 = 1) \\ \text{Eq. } 12.10(v_1 = 1) & \text{Eq. } 12.10(\theta_1 = 1) & \text{Eq. } 12.10(v_2 = 1) & \text{Eq. } 12.10(\theta_2 = 1) \end{array}\right] \end{array} \tag{13.3}$$

25 The stiffness matrix of the beam element (neglecting shear and axial deformation) will thus be

$$\boxed{[\mathbf{k}^b] = \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \left[\begin{array}{cccc} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{array}\right] \end{array}} \tag{13.4}$$

26 We note that this is identical to Eq.12.14

$$\left\{ \begin{array}{c} V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \right\} = \underbrace{\begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \left[\begin{array}{cccc} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{array}\right] \end{array}}_{\mathbf{k}^{(e)}} \left\{ \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array} \right\} \tag{13.5}$$

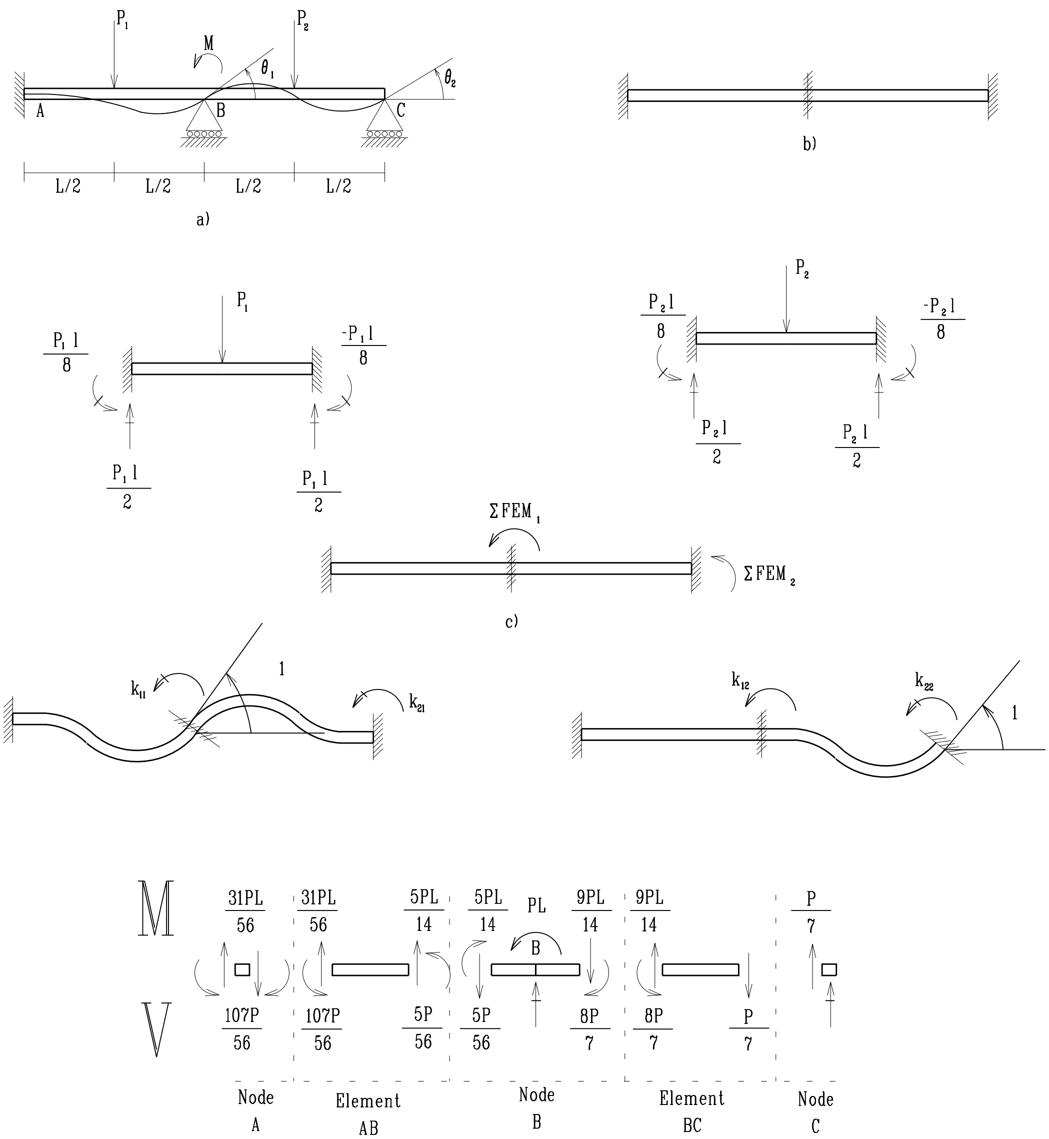

Figure 13.7: Problem with 2 Global d.o.f. θ_1 and θ_2

```
% Solve for the Displacements in meters and radians
Displacements=inv(Ktt)*P'
% Extract Kut
Kut=Kaug(4:9,1:3);
% Compute the Reactions and do not forget to add fixed end actions
Reactions=Kut*Displacements+FEA_Rest'
% Solve for the internal forces and do not forget to include the fixed end actions
dis_global(:,:,1)=[0,0,0,Displacements(1:3)'];
dis_global(:,:,2)=[Displacements(1:3)',0,0,0];
for elem=1:2
   dis_local=Gamma(:,:,elem)*dis_global(:,:,elem)';
   int_forces=k(:,:,elem)*dis_local+fea(1:6,elem)
end

function [k,K,Gamma]=stiff(EE,II,A,i,j)
% Determine the length
L=sqrt((j(2)-i(2))^2+(j(1)-i(1))^2);
% Compute the angle theta (carefull with vertical members!)
if(j(1)-i(1))~=0
   alpha=atan((j(2)-i(2))/(j(1)-i(1)));
else
   alpha=-pi/2;
end
% form rotation matrix Gamma
Gamma=[
cos(alpha)  sin(alpha)  0  0          0          0;
-sin(alpha) cos(alpha)  0  0          0          0;
0           0           1  0          0          0;
0           0           0  cos(alpha) sin(alpha) 0;
0           0           0 -sin(alpha) cos(alpha) 0;
0           0           0  0          0          1];
% form element stiffness matrix in local coordinate system
EI=EE*II;
EA=EE*A;
k=[EA/L,         0,         0, -EA/L,          0,         0;
   0,    12*EI/L^3,  6*EI/L^2,     0, -12*EI/L^3,  6*EI/L^2;
   0,     6*EI/L^2,    4*EI/L,     0,  -6*EI/L^2,    2*EI/L;
 -EA/L,          0,         0,  EA/L,          0,         0;
     0, -12*EI/L^3, -6*EI/L^2,     0,  12*EI/L^3, -6*EI/L^2;
     0,   6*EI/L^2,    2*EI/L,     0,  -6*EI/L^2,    4*EI/L];
% Element stiffness matrix in global coordinate system
K=Gamma'*k*Gamma;
```

This simple proigram will produce the following results:

```
Displacements =

    0.0010
   -0.0050
```

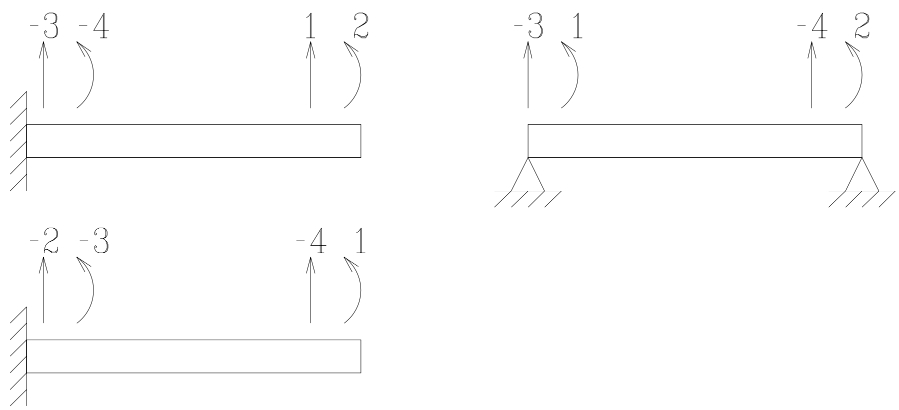

Figure 13.14: ID Values for Simple Beam

2. The *structure* stiffness matrix is assembled

$$\mathbf{K} = \begin{array}{c} \\ 1 \\ 2 \\ -3 \\ -4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & -3 & -4 \end{array} \\ \left[\begin{array}{cccc} 12EI/L^2 & -6EI/L^2 & -12EI/L^3 & -6EI/L^2 \\ -6EI/L^2 & 4EI/L & 6EI/L^2 & 2EI/L \\ -12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\ -6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L \end{array}\right] \end{array}$$

3. The global matrix can be rewritten as

$$\left\{\begin{array}{c} -P\surd \\ 0\surd \\ \hline R_3? \\ R_4? \end{array}\right\} = \left[\begin{array}{cc|cc} 12EI/L^2 & -6EI/L^2 & -12EI/L^3 & -6EI/L^2 \\ -6EI/L^2 & 4EI/L & 6EI/L^2 & 2EI/L \\ \hline -12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\ -6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L \end{array}\right] \left\{\begin{array}{c} \Delta_1? \\ \theta_2? \\ \hline \Delta_3\surd \\ \theta_4\surd \end{array}\right\}$$

4. $\mathbf{K}_{tt}$ is inverted (or actually decomposed) and stored in the same global matrix

$$\left[\begin{array}{cc|cc} \boxed{L^3/3EI} & \boxed{L^2/2EI} & -12EI/L^3 & -6EI/L^2 \\ \boxed{L^2/2EI} & \boxed{L/EI} & 6EI/L^2 & 2EI/L \\ \hline -12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\ -6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L \end{array}\right]$$

5. Next we compute the equivalent load, $\mathbf{P}'_t = \mathbf{P}_t - \mathbf{K}_{tu}\mathbf{\Delta}_u$, and overwrite $\mathbf{P}_t$ by $\mathbf{P}'_t$

$$\begin{aligned} \mathbf{P}_t - \mathbf{K}_{tu}\mathbf{\Delta}_u &= \left\{\begin{array}{c} \boxed{-P} \\ \boxed{0} \\ \hline 0 \\ 0 \end{array}\right\} - \left[\begin{array}{cc|cc} L^3/3EI & L^2/2EI & \boxed{-12EI/L^3} & \boxed{-6EI/L^2} \\ L^2/2EI & L/EI & \boxed{6EI/L^2} & \boxed{2EI/L} \\ \hline -12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\ -6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L \end{array}\right] \left\{\begin{array}{c} -P \\ 0 \\ \hline \boxed{0} \\ \boxed{0} \end{array}\right\} \\ &= \left\{\begin{array}{c} \boxed{-P} \\ \boxed{0} \\ \hline 0 \\ 0 \end{array}\right\} \end{aligned}$$

5. We compute the equivalent load, $\mathbf{P}'_t = \mathbf{P}_t - \mathbf{K}_{tu}\boldsymbol{\Delta}_u$, and overwrite $\mathbf{P}_t$ by $\mathbf{P}'_t$

$$\mathbf{P}_t - \mathbf{K}_{tu}\boldsymbol{\Delta}_u = \left\{\begin{array}{c}\boxed{0}\\ \boxed{M}\\ \hline 0\\ 0\end{array}\right\} - \left[\begin{array}{cc|cc} L^3/3EI & -L/6EI & \boxed{6EI/L^2} & \boxed{-6EI/L^2}\\ -L/6EI & L/3EI & \boxed{6EI/L^2} & \boxed{-6EI/L^2}\\ \hline 6EI/L^2 & 6EI/L^2 & 12EI/L^3 & -12EI/L^3\\ -6EI/L^2 & -6EI/L^2 & -12EI/L^3 & 12EI/L^3\end{array}\right]\left\{\begin{array}{c}0\\ M\\ \hline \boxed{0}\\ \boxed{0}\end{array}\right\}$$

$$= \left\{\begin{array}{c}\boxed{0}\\ \boxed{M}\\ \hline 0\\ 0\end{array}\right\}$$

6. Solve for the displacements, $\boldsymbol{\Delta}_t = \mathbf{K}_{tt}^{-1}\mathbf{P}'_t$, and overwrite $\mathbf{P}_t$ by $\boldsymbol{\Delta}_t$

$$\left\{\begin{array}{c}\boxed{\theta_1}\\ \boxed{\theta_2}\\ \hline 0\\ 0\end{array}\right\} = \left[\begin{array}{cc|cc} \boxed{L^3/3EI} & \boxed{-L/6EI} & 6EI/L^2 & -6EI/L^2\\ \boxed{-L/6EI} & \boxed{L/3EI} & 6EI/L^2 & -6EI/L^2\\ \hline 6EI/L^2 & 6EI/L^2 & 12EI/L^3 & -12EI/L^3\\ -6EI/L^2 & -6EI/L^2 & -12EI/L^3 & 12EI/L^3\end{array}\right]\left\{\begin{array}{c}\boxed{0}\\ \boxed{M}\\ \hline 0\\ 0\end{array}\right\}$$

$$= \left\{\begin{array}{c}\boxed{-ML/6EI}\\ \boxed{ML/3EI}\\ \hline 0\\ 0\end{array}\right\}$$

7. Solve for the reactions, $\mathbf{R}_t = \mathbf{K}_{ut}\boldsymbol{\Delta}_{tt} + \mathbf{K}_{uu}\boldsymbol{\Delta}_u$, and overwrite $\boldsymbol{\Delta}_u$ by $\mathbf{R}_u$

$$\left\{\begin{array}{c}-ML/6EI\\ ML/3EI\\ \hline \boxed{R_1}\\ \boxed{R_2}\end{array}\right\} = \left[\begin{array}{cc|cc} L^3/3EI & -L/6EI & 6EI/L^2 & -6EI/L^2\\ -L/6EI & L/3EI & 6EI/L^2 & -6EI/L^2\\ \hline \boxed{6EI/L^2} & \boxed{6EI/L^2} & \boxed{12EI/L^3} & \boxed{-12EI/L^3}\\ \boxed{-6EI/L^2} & \boxed{-6EI/L^2} & \boxed{-12EI/L^3} & \boxed{12EI/L^3}\end{array}\right]\left\{\begin{array}{c}\boxed{-ML/6EI}\\ \boxed{ML/3EI}\\ \hline \boxed{0}\\ \boxed{0}\end{array}\right\}$$

$$= \left\{\begin{array}{c}-ML/6EI\\ ML/3EI\\ \hline \boxed{M/L}\\ \boxed{-M/L}\end{array}\right\}$$

Cantilivered Beam/Initial Displacement and Concentrated Moment

1. The *element* stiffness matrix is

$$\mathbf{k} = \begin{array}{c|cccc} & -2 & -3 & -4 & 1\\ \hline -2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2\\ -3 & 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L\\ -4 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2\\ 1 & 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L\end{array}$$

2. The *structure* stiffness matrix is assembled

$$\mathbf{K} = \begin{array}{c|cccc} & 1 & -2 & -3 & -4\\ \hline 1 & 4EI/L & 6EI/L^2 & 2EI/L & -6EI/L^2\\ -2 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3\\ -3 & 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2\\ -4 & -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3\end{array}$$

3. The global matrix can be rewritten as

$$\left\{\begin{array}{c}M\surd\\ \hline R_2?\\ R_3?\\ R_4?\end{array}\right\} = \left[\begin{array}{c|ccc} 4EI/L & 6EI/L^2 & 2EI/L & -6EI/L^2\\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3\\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2\\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3\end{array}\right]\left\{\begin{array}{c}\theta_1?\\ \hline \Delta_2\surd\\ \theta_3\surd\\ \Delta_4\surd\end{array}\right\}$$