Chapter 8

Trip distribution

8.1 Overview

The decision to travel for a given purpose is called trip generation. These generated trips from each zone is then distributed to all other zones based on the choice of destination. This is called trip distribution which forms the second stage of travel demand modeling. There are a number of methods to distribute trips among destinations; and two such methods are growth factor model and gravity model. Growth factor model is a method which respond only to relative growth rates at origins and destinations and this is suitable for short-term trend extrapolation. In gravity model, we start from assumptions about trip making behavior and the way it is influenced by external factors. An important aspect of the use of gravity models is their calibration, that is the task of fixing their parameters so that the base year travel pattern is well represented by the model.

8.2 Definitions and notations

8.2.1 Trip matrix

The trip pattern in a study area can be represented by means of a trip matrix or origin-destination (O-D)matrix. This is a two dimensional array of cells where rows and columns represent each of the zones in the study area. The notation of the trip matrix is given in figure 1.1.

The cells of each row i contain the trips originating in that zone which have as destinations the zones in the corresponding columns. T_{ij} is the number of trips between origin i and destination j. O_i is the total number

Zones	1	2	...	j	...	n	$O_i = \Sigma_j T_{ij}$
1	T_{11}	T_{12}	...	T_{1j}	...	T_{1n}	O_1
2	T_{21}	T_{22}	...	T_{2j}	...	T_{2n}	O_2
$\vdots$	...	...	...	...	...	...	$\vdots$
i	T_{i1}	T_{i2}	...	T_{ij}	...	T_{in}	O_i
$\vdots$	...	...	...	...	...	...	$\vdots$
n	T_{ni}	T_{n2}	...	T_{nj}	...	T_{nn}	O_n
$D_j = \Sigma_i T_{ij}$	D_1	D_2	...	D_j	...	D_n	$T = \Sigma_{ij} T_{ij}$

Figure 8:1: Notation of a trip matrix

of trips between originating in zone i and D_j is the total number of trips attracted to zone j. The sum of the trips in a row should be equal to the total number of trips emanating from that zone. The sum of the trips in a column is the number of trips attracted to that zone. These two constraints can be represented as: $\Sigma_j T_{ij} = O_i$ $\Sigma_i T_{ij} = D_j$ If reliable information is available to estimate both O_i and D_j, the model is said to be doubly constrained. In some cases, there will be information about only one of these constraints, the model is called singly constrained.

8.2.2 Generalized cost

One of the factors that influences trip distribution is the relative travel cost between two zones. This cost element may be considered in terms of distance, time or money units. It is often convenient to use a measure combining all the main attributes related to the dis-utility of a journey and this is normally referred to as the generalized cost of travel. This can be represented as

$$c_{ij} = a_1 t_{ij}^v + a_2 t_{ij}^w + a_3 t_{ij}^t + a_4 t_{nij} + a_5 F_{ij} + a_6 \phi_j + \delta \tag{8.1}$$

where t_{ij}^v is the in-vehicle travel time between i and j, t_{ij}^w is the walking time to and from stops, t_{ij}^t is the waiting time at stops, F_{ij} is the fare charged to travel between i and j, ϕ_j is the parking cost at the destination, and δ is a parameter representing comfort and convenience, and a_1,a_2,.... are the weights attached to each element of cost function.

8.3 Growth factor methods

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8.3.1 Uniform growth factor

If the only information available is about a general growth rate for the whole of the study area, then we can only assume that it will apply to each cell in the matrix, that is a uniform growth rate. The equation can be written as:

$$T_{ij} = f \times t_{ij} \tag{8.2}$$

where f is the uniform growth factor t_{ij} is the previous total number of trips, T_{ij} is the expanded total number of trips. Advantages are that they are simple to understand, and they are useful for short-term planning. Limitation is that the same growth factor is assumed for all zones as well as attractions.

8.3.2 Example

Trips originating from zone 1,2,3 of a study area are 78,92 and 82 respectively and those terminating at zones 1,2,3 are given as 88,96 and 78 respectively. If the growth factor is 1.3 and the cost matrix is as shown below, find the expanded origin-constrained growth trip table.

	1	2	3	o_i
1	20	30	28	78
2	36	32	24	92
3	22	34	26	82
d_j	88	96	78	252

Solution

Given growth factor = 1.3, Therefore, multiplying the growth factor with each of the cells in the matrix gives the solution as shown below.

	1	2	3	O_i
1	26	39	36.4	101.4
2	46.8	41.6	31.2	119.6
3	28.6	44.2	33.8	106.2
D_j	101.4	124.8	101.4	327.6

8.3.3 Doubly constrained growth factor model

When information is available on the growth in the number of trips originating and terminating in each zone, we know that there will be different growth rates for trips in and out of each zone and consequently having two sets of growth factors for each zone. This implies that there are two constraints for that model and such a model is called doubly constrained growth factor model. One of the methods of solving such a model is given by Furness who introduced *balancing factors* a_i and b_j as follows:

$$T_{ij} = t_{ij} \times a_i \times b_j \tag{8.3}$$

In such cases, a set of intermediate correction coefficients are calculated which are then appropriately applied to cell entries in each row or column. After applying these corrections to say each row, totals for each column are calculated and compared with the target values. If the differences are significant, correction coefficients are calculated and applied as necessary. The procedure is given below:

1. Set $b_j = 1$
2. With b_j solve for a_i to satisfy trip generation constraint.
3. With a_i solve for b_j to satisfy trip attraction constraint.
4. Update matrix and check for errors.
5. Repeat steps 2 and 3 till convergence.

Here the error is calculated as: $E = \Sigma|O_i - O_i^1| + \Sigma|D_j - D_j^1|$ where O_i corresponds to the actual productions from zone i and O_i^1 is the calculated productions from that zone. Similarly D_j are the actual attractions from the zone j and D_j^1 are the calculated attractions from that zone.

8.3.4 Advantages and limitations of growth factor model

The advantages of this method are:

1. Simple to understand.
2. Preserve observed trip pattern.
3. useful in short term-planning.

The limitations are:

1. Depends heavily on the observed trip pattern.

2. It cannot explain unobserved trips.

3. Do not consider changes in travel cost.

4. Not suitable for policy studies like introduction of a mode.

Example The base year trip matrix for a study area consisting of three zones is given below

	1	2	3	o_i
1	20	30	28	78
2	36	32	24	92
3	22	34	26	82
d_j	88	96	78	252

The productions from the zone 1,2 and 3 for the horizon year is expected to grow to 98, 106, and 122 respectively. The attractions from these zones are expected to increase to 102, 118, 106 respectively. Compute the trip matrix for the horizon year using doubly constrained growth factor model using Furness method

Solution The sum of the attractions in the horizon year, i.e. ΣO_i = 98+106+122 = 326. The sum of the productions in the horizon year, i.e. ΣD_j = 102+118+106 = 326. They both are found to be equal. Therefore we can proceed. The first step is to fix $b_j = 1$, and find balancing factor a_i. $a_i = O_i/o_i$, then find $T_{ij} = a_i \times t_{ij}$

So $a_1 = 98/78 = 1.26$

$a_2 = 106/92 = 1.15$

$a_3 = 122/82 = 1.49$ Further $T_{11} = t_{11} \times a_1 = 20 \times 1.26 = 25.2$. Similary $T_{12} = t_{12} \times a_2 = 36 \times 1.15 = 41.4$. etc. Multiplying a_1 with the first row of the matrix, a_2 with the second row and so on, matrix obtained is as shown below.

	1	2	3	o_i
1	25.2	37.8	35.28	98
2	41.4	36.8	27.6	106
3	32.78	50.66	38.74	122
d_j^1	99.38	125.26	101.62	
D_j	102	118	106	

Also $d_j^1 = 25.2 + 41.4 + 32.78 = 99.38$

In the second step, find $b_j = D_j/d_j^1$ and $T_{ij} = t_{ij} \times b_j$. For example $b_1 = 102/99.38 = 1.03$, $b_2 = 118/125.26 = 0.94$ etc.,$T_{11} = t_1 1 \times b_1 = 25.2 \times 1.03 = 25.96$ etc. Also $O_i^1 = 25.96 + 35.53 + 36.69 = 98.18$. The matrix is as shown below:

	1	2	3	o_i	O_i
1	25.96	35.53	36.69	98.18	98
2	42.64	34.59	28.70	105.93	106
3	33.76	47.62	40.29	121.67	122
b_j	1.03	0.94	1.04		
D_j	102	118	106		

	1	2	3	O_i^1	O_i
1	25.96	35.53	36.69	98.18	98
2	42.64	34.59	28.70	105.93	106
3	33.76	47.62	40.29	121.67	122
d_j	102.36	117.74	105.68	325.78	
D_j	102	118	106	326	

Therefore error can be computed as ; $Error = \Sigma|O_i - O_i^1| + \Sigma|D_j - d_j|$

Error = —98.18-98—+—105.93-106—+—121.67-122—+—102.36-102—+—117.74-118—+—105.68-106— = 1.32

8.4 Gravity model

This model originally generated from an analogy with Newton's gravitational law. Newton's gravitational law says, $F = GM_1M_2/d_2$ Analogous to this, $T_{ij} = CO_iD_j/c_{ij^n}$ Introducing some balancing factors, $T_{ij} = A_iO_iB_jD_jf(c_{ij})$ where A_i and B_j are the balancing factors, $f(c_{ij})$ is the generalized function of the travel cost. This function is called *deterrence function* because it represents the disincentive to travel as distance (time) or cost increases. Some of the versions of this function are:

$$f(c_{ij}) = e^{-\beta c_{ij}} \tag{8.4}$$

$$f(c_{OJ}) = c_{ij}^{-n} \tag{8.5}$$

$$f(c_{ij}) = c_{ij}^{-n} \times e^{-\beta c_{ij}} \tag{8.6}$$

The first equation is called the exponential function, second one is called power function where as the third one is a combination of exponential and power function. The general form of these functions for different values of their parameters is as shown in figure.

As in the growth factor model, here also we have singly and doubly constrained models. The expression $T_{ij} = A_iO_iB_jD_jf(c_{ij})$ is the classical version of the doubly constrained model. Singly constrained versions can be produced by making one set of balancing factors A_i or B_j equal to one. Therefore we can treat singly constrained model as a special case which can be derived from doubly constrained models. Hence we will limit our discussion to doubly constrained models.

As seen earlier, the model has the functional form, $T_{ij} = A_iO_iB_jD_jf(c_{ij})$

$$\Sigma_iT_{ij} = \Sigma_iA_iO_iB_jD_jf(c_{ij}) \tag{8.7}$$

But

$$\Sigma_iT_{ij} = D_j \tag{8.8}$$

Therefore,

$$D_j = B_jD_j\Sigma_iA_iO_if(c_{ij}) \tag{8.9}$$

From this we can find the balancing factor B_j as

$$B_j = 1/\Sigma_iA_iO_if(c_{ij}) \tag{8.10}$$

B_j depends on A_i which can be found out by the following equation:

$$A_i = 1/\Sigma_jB_jD_jf(c_{ij}) \tag{8.11}$$

i	j	B_j	D_J	$f(c_{ij})$	$B_j D_j f(c_{ij})$	$\Sigma B_j D_j f(c_{ij})$	$A_i = \frac{1}{\Sigma B_j D_j f(c_{ij})}$
	1	1.0	102	1.0	102.00		
1	2	1.0	118	0.69	81.42	216.28	0.00462
	3	1.0	106	0.31	32.86		
	1	1.0	102	0.69	70.38		
2	2	1.0	118	1.0	118	235.02	0.00425
	3	1.0	106	0.44	46.64		
	1	1.0	102	0.31	31.62		
3	2	1.0	118	0.44	51.92	189.54	0.00527
	3	1.0	106	1.00	106		

We can see that both A_i and B_j are interdependent. Therefore, through some iteration procedure similar to that of Furness method, the problem can be solved. The procedure is discussed below:

1. Set $B_j = 1$, find A_i using equation 1.11
2. Find B_j using equation 1.10
3. Compute the error as $E = \Sigma|O_i - O_i^1| + \Sigma|D_j - D_j^1|$ where O_i corresponds to the actual productions from zone i and O_i^1 is the calculated productions from that zone. Similarly D_j are the actual attractions from the zone j and D_j^1 are the calculated attractions from that zone.
4. Again set $B_j = 1$ and find A_i, also find B_j. Repeat these steps until the convergence is achieved.

Example The productions from zone 1, 2 and 3 are 98, 106, 122 and attractions to zone 1,2 and 3 are 102, 118, 106. The function $f(c_{ij})$ is defined as $f(c_{ij}) = 1/c_{ij}^2$ The cost matrix is as shown below

$$\begin{bmatrix} 1.0 & 1.2 & 1.8 \\ 1.2 & 1.0 & 1.5 \\ 1.8 & 1.5 & 1.0 \end{bmatrix} \tag{8.12}$$

Solution

The second step is to find B_j. This can be found out as $B_j = 1/\Sigma A_i O_i f(c_{ij})$, where A_i is obtained from the previous step.

The function $f(c_{ij})$ can be written in the matrix form as:

$$\begin{bmatrix} 1.0 & 0.69 & 0.31 \\ 0.69 & 1.0 & 0.44 \\ 0.31 & 0.44 & 1.0 \end{bmatrix} \tag{8.13}$$

Then T_{ij} can be computed using the formula

$$T_{ij} = A_i O_i B_j D_j f(c_{ij}) \tag{8.14}$$

For eg, $T_{11} = 102 \times 1.0397 \times 0.00462 \times 98 \times 1 = 48.01$. O_i is the actual productions from the zone and O_i^1 is the computed ones. Similar is the case with attractions also.

j	i	A_i	O_i	$f(c_{ij})$	$A_iO_if(c_{ij})$	$\Sigma A_iO_if(c_{ij})$	$B_j = 1/\Sigma A_iO_if(c_{ij})$
	1	0.00462	98	1.0	0.4523		
1	2	0.00425	106	0.694	0.3117	0.9618	1.0397
	3	0.00527	122	0.308	0.1978		
	1	0.00462	98	0.69	0.3124		
2	2	0.00425	106	1.0	0.4505	1.0458	0.9562
	3	0.00527	122	0.44	0.2829		
	1	0.00462	98	0.31	0.1404		
3	2	0.00425	106	0.44	0.1982	0.9815	1.0188
	3	0.00527	122	1.00	0.6429		

	1	2	3	A_i	O_i	O_i^1
1	48.01	35.24	15.157	0.00462	98	98.407
2	32.96	50.83	21.40	0.00425	106	105.19
3	21.14	31.919	69.43	0.00527	122	122.489
B_j	1.0397	0.9562	1.0188			
D_j	102	118	106			
D_j^1	102.11	117.989	105.987			

O_i is the actual productions from the zone and O_i^1 is the computed ones. Similar is the case with attractions also.

Therefore error can be computed as ; $Error = \Sigma|O_i - O_i^1| + \Sigma|D_j - D_j^1|$ $Error = |98 - 98.407| + |106 - 105.19| + |122 - 122.489| + ||102 - 102.11| + |118 - 117.989| + |106 - 105.987| = 2.03$

8.5 Summary

8.6 Problems

1. The trip productions from zones 1, 2 and 3 are 110, 122 and 114 respectively and the trip attractions to these zones are 120,134 and 108 respectively. The cost matrix is given below. The function $f(c_{ij}) = \frac{1}{c_{ij}}$

$$\begin{bmatrix} 1.0 & 1.2 & 1.8 \\ 1.2 & 1.0 & 1.5 \\ 1.8 & 1.5 & 1.0 \end{bmatrix}$$

   Compute the trip matrix using doubly constrained gravity model. Provide one complete iteration. Solution

i	j	B_j	D_J	$f(c_{ij})$	$B_jD_jf(c_{ij})$	$\Sigma B_jD_jf(c_{ij})$	$A_i = \frac{1}{\Sigma B_jD_jf(c_{ij})}$
	1	1.0	120	1.0	120.00		
1	2	1.0	108	0.833	89.964	275.454	0.00363
	3	1.0	118	0.555	65.49		
	1	1.0	120	0.833	99.96		
2	2	1.0	108	1.0	108	286.66	0.00348
	3	1.0	118	0.667	78.706		
	1	1.0	120	0.555	66.60		
3	2	1.0	108	0.667	72.036	256.636	0.00389
	3	1.0	118	1.00	118		

The second step is to find B_j. This can be found out as $B_j = 1/\Sigma A_iO_if(c_{ij})$, where A_i is obtained from the previous step.

j	i	A_i	O_i	$f(c_{ij})$	$A_iO_if(c_{ij})$	$\Sigma A_iO_if(c_{ij})$	$B_j = 1/\Sigma A_iO_if(c_{ij})$
	1	0.00363	110	1.0	0.3993		
1	2	0.00348	122	0.833	0.3536	0.9994	1.048
	3	0.00389	114	0.555	0.2465		
	1	0.00363	110	0.833	0.3326		
2	2	0.00348	122	1.0	0.4245	1.05	0.9494
	3	0.00389	114	0.667	0.2962		
	1	0.00363	110	0555	0.2216		
3	2	0.00348	122	0.667	0.2832	0.9483	1.054
	3	0.00389	114	1.00	0.44346		

The function $f(c_{ij})$ can be written in the matrix form as:

$$\begin{bmatrix} 1.0 & 0.833 & 0.555 \\ 0.833 & 1.0 & 0.667 \\ 0.555 & 0.667 & 1.0 \end{bmatrix} \tag{8.15}$$

Then T_{ij} can be computed using the formula

$$T_{ij} = A_iO_iB_jD_jf(c_{ij}) \tag{8.16}$$

For eg, $T_{11} = 102 \times 1.0397 \times 0.00462 \times 98 \times 1 = 48.01$. O_i is the actual productions from the zone and O_i^1 is the computed ones. Similar is the case with attractions also.

	1	2	3	A_i	O_i	O_i^1
1	47.916	34.10	27.56	0.00363	110	109.57
2	42.43	43.53	35.21	0.00348	122	121.17
3	29.53	30.32	55.15	0.00389	114	115
B_j	1.00	0.9494	1.054			
D_j	120	108	118			
D_j^1	119.876	107.95	117.92			

O_i is the actual productions from the zone and O_i^1 is the computed ones. Similar is the case with attractions also.

Therefore error can be computed as ; $Error = \Sigma|O_i - O_i^1| + \Sigma|D_j - D_j^1|$ $Error = |110 - 109.57| + |122 - 121.17| + |114 - 115| + |120 - 119.876 + |108 - 107.95| + |118 - 117.92| = 2.515$