Chapter 1: Theory of Vibrations

Subject: Earthquake Engineering Program: B.Tech Civil Engineering (Elective)

1.1 Introduction

Understanding vibrations is fundamental to earthquake engineering, as ground motion during an earthquake induces vibratory responses in structures. The Theory of Vibrations deals with the analysis of dynamic behavior of systems subjected to time-varying disturbances. In civil engineering, it is particularly relevant for designing buildings and infrastructure that can withstand seismic events.

Vibrations may be induced by various sources, such as machinery, wind, traffic, or seismic activity. These vibrations can be detrimental if not properly accounted for in design, potentially leading to serviceability issues or even structural failure. This chapter lays the foundation for analyzing such dynamic effects in structures.

1.2 Basic Terminologies and Concepts

1.2.1 Vibration

Vibration is defined as the oscillatory motion of a body about an equilibrium position. It can be:

- **Free vibration**: Occurs without external force after an initial disturbance.
- **Forced vibration**: Occurs due to continuous external excitation.

1.2.2 Types of Vibratory Systems

- **Single Degree of Freedom (SDOF) System**: A system that requires only one coordinate to describe its motion.
- **Multiple Degrees of Freedom (MDOF) System**: Systems requiring two or more independent coordinates.
- **Continuous Systems**: Systems like beams or plates with infinite degrees of freedom.

1.2.3 Key Parameters of Vibration

- **Displacement (x)**: The distance moved from equilibrium.
- **Velocity (ẋ)**: Rate of change of displacement.
- **Acceleration (ẍ)**: Rate of change of velocity.
- **Mass (m)**: Inertia of the vibrating body.
- **Stiffness (k)**: Resistance to deformation.
- **Damping (c)**: Energy dissipation mechanism.

1.3 Free Vibration of SDOF Systems

Consider a mass-spring system with no damping. The equation of motion is:

$$m\ddot{x} + kx = 0$$

This is a second-order homogeneous differential equation. Its solution is:

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

Where:

- $\omega_n = \sqrt{\frac{k}{m}}$: Natural circular frequency (rad/s)
- $f_n = \frac{\omega_n}{2\pi}$: Natural frequency (Hz)
- A, B: Constants based on initial conditions

The system oscillates indefinitely at its natural frequency in absence of damping.

1.4 Damped Free Vibration

When damping is included:

$$m\ddot{x} + c\dot{x} + kx = 0$$

The nature of the solution depends on the damping ratio:

$$\zeta = \frac{c}{2\sqrt{mk}}$$

Cases of Damped Vibration:

- **Underdamped ($\zeta < 1$)**: Oscillatory motion with exponential decay

- **Critically damped ($\zeta = 1$)**: Fastest return to equilibrium without oscillation
- **Overdamped ($\zeta > 1$)**: No oscillation; slow return to equilibrium

For underdamped systems:

$$x(t) = e^{-\zeta \omega_n t}(A \cos \omega_d t + B \sin \omega_d t)$$

Where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$ is the damped natural frequency.

1.5 Forced Vibration of SDOF Systems

For a harmonic external force $F(t) = F_0 \sin(\omega t)$, the equation becomes:

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t)$$

The solution has two parts:

- **Transient response**: Decays with time
- **Steady-state response**: Dominates at long times

Steady-state amplitude is given by:

$$X = \frac{F_0}{\sqrt{¿¿¿}}$$

This amplitude peaks near the natural frequency, causing **resonance** when $\omega \approx \omega_n$.

1.6 Vibration Response Parameters

- **Amplitude**: Maximum displacement from the mean position.
- **Phase angle**: The lag between the excitation and response.
- **Frequency response**: Plot of amplitude vs frequency.
- **Logarithmic decrement**: Measures damping in underdamped systems:

$$\delta = \ln\left(\frac{x(t)}{x(t+T)}\right)$$

Where T is the period of vibration.

1.7 Response of Structures to Ground Motion

Earthquakes apply dynamic forces at the base of a structure, modeled by base excitation:

$$m\ddot{x}+c(\dot{x}-\dot{y})+k(x-y)=0$$

Where $y(t)$ is the ground displacement.

Relative displacement $u=x-y$, leads to:

$$m\ddot{u}+c\dot{u}+ku=-m\ddot{y}$$

This model forms the basis for **response spectrum analysis**.

1.8 Multi-Degree of Freedom (MDOF) Systems

1.8.1 Equations of Motion

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F(t)\}$$

Where:

- $[M]$: Mass matrix
- $[C]$: Damping matrix
- $[K]$: Stiffness matrix
- $\{x\}$: Displacement vector
- $\{F(t)\}$: Force vector

1.8.2 Mode Shapes and Natural Frequencies

Solving the **eigenvalue problem**:

$$([K]-\omega^2[M])\{\phi\}=0$$

Gives natural frequencies ω_n and mode shapes $\{\phi\}$.

Each mode vibrates independently in **modal analysis**, and total response is a superposition of all modal responses.

1.9 Damping in Structures

Damping is vital for reducing seismic response.

Types of Damping:

- **Viscous damping**: Force proportional to velocity
- **Coulomb (frictional) damping**
- **Hysteretic damping**: Energy lost in cyclic loading
- **Structural damping**: Inherent material behavior

Real-world damping is often approximated as viscous in analytical models.

1.10 Numerical Methods for Vibration Analysis

When closed-form solutions are infeasible (especially for MDOF systems or nonlinear problems), numerical methods are used.

1.10.1 Finite Difference Method (FDM)

Discretizes time and solves equations step-by-step.

1.10.2 Newmark's Method

A common time integration method used in seismic analysis.

$$\ddot{x}_{n+1} = \frac{1}{\beta \Delta t^2}\left(x_{n+1} - x_n - \Delta t\, \dot{x}_n\right) - \frac{1 - 2\beta}{2\beta}\ddot{x}_n$$

1.10.3 Mode Superposition Method

Solves MDOF problems by transforming to modal coordinates, solving each SDOF mode, and combining responses.

1.11 Vibration Isolation and Control

To reduce vibration impact:

- **Base Isolation**: Separates structure from ground shaking.
- **Tuned Mass Dampers (TMDs)**: Add mass tuned to cancel vibration.
- **Active Control Systems**: Use sensors and actuators to suppress motion.

These strategies are critical in earthquake-resistant design.

1.12 Resonance and Its Implications in Earthquake Engineering

Resonance is a critical phenomenon where the frequency of external excitation (such as ground motion) matches the natural frequency of a structure, resulting in large amplitude oscillations.

1.12.1 Conditions for Resonance

- Resonance occurs when:

$$\omega = \omega_n$$

- At resonance, the amplitude becomes maximum if damping is low.
- Structures with poor damping and natural frequencies within the earthquake excitation band are highly vulnerable.

1.12.2 Structural Response During Resonance

- Amplified displacements and accelerations.
- Excessive inter-storey drift, leading to:
  - Cracking in structural/non-structural elements.
  - Failure of connections and joints.
  - Total collapse in extreme cases.

1.12.3 Mitigation of Resonance Effects

- Shift natural frequency by changing stiffness or mass.
- Introduce sufficient damping.
- Avoid resonance frequency range during structural design.

1.13 Earthquake Excitation Characteristics

Understanding the characteristics of ground motion helps predict structural response.

1.13.1 Important Parameters of Ground Motion

- **Peak Ground Acceleration (PGA)**: Maximum acceleration experienced.
- **Duration**: Time span of significant shaking.
- **Frequency content**: Distribution of energy across frequencies.
- **Time history**: Record of acceleration over time.
- **Spectral content**: Useful for response spectrum analysis.

1.13.2 Frequency Ranges of Earthquake Motions

- Low-rise buildings (1–3 storeys): Natural frequency ~2–6 Hz
- Medium-rise buildings (4–7 storeys): ~1–3 Hz
- High-rise buildings (8+ storeys): ~0.2–1.0 Hz

If the earthquake contains dominant energy in the same frequency range as a structure, resonance risk increases.

1.14 Dynamic Amplification Factor (DAF)

Dynamic Amplification Factor quantifies how much greater the response is under dynamic loading compared to static.

$$DAF = \frac{\text{Dynamic Displacement}}{\text{Static Displacement}}$$

For undamped systems:

$$DAF = \frac{1}{¿ 1 - ¿¿}$$

Inclusion of damping reduces DAF significantly, emphasizing the importance of incorporating damping into design.

1.15 Seismic Design Considerations Based on Vibration Theory

Structural codes and design methodologies incorporate vibration theory into seismic-resistant design:

1.15.1 Code-Based Requirements

- **IS 1893 (Part 1):** Criteria for earthquake-resistant design of structures (India)
- Requires:
  - Estimation of natural periods.
  - Use of response spectra.
  - Consideration of damping.
  - Dynamic analysis for taller and irregular structures.

1.15.2 Design Based on Dynamic Characteristics

- Avoid tuning natural frequency to predominant ground motion frequency.
- Evaluate torsional irregularities in MDOF systems.
- Apply modal analysis for dynamic load combinations.

1.16 Experimental Modal Analysis and Structural Health Monitoring

Laboratory and field experiments help determine real dynamic characteristics.

1.16.1 Experimental Modal Analysis (EMA)

- Uses controlled excitation and sensors to determine:
  - Natural frequencies
  - Mode shapes
  - Damping ratios

1.16.2 Structural Health Monitoring (SHM)

- Sensors placed in buildings to continuously monitor vibration.
- Tracks changes in dynamic properties which may indicate:
  - Structural damage
  - Fatigue
  - Deterioration

Used in bridges, high-rise buildings, and critical infrastructure.

1.17 Role of Computational Tools in Vibration Analysis

Modern earthquake engineering relies on simulation for vibration analysis:

1.17.1 Finite Element Method (FEM)

- Breaks complex structures into discrete elements.
- Solves MDOF vibration problems numerically.

1.17.2 Software for Vibration and Seismic Analysis

- **ETABS**, **SAP2000**, **STAAD.Pro**, **ANSYS**, **OpenSees**
- Can perform:
  - Modal analysis
  - Time history analysis
  - Response spectrum analysis
  - Nonlinear dynamic simulation

These tools allow accurate prediction of real-world behavior under seismic loads.

1.18 Vibration Control Devices in Modern Structures

Advancements in materials and mechanics have introduced sophisticated control systems:

1.18.1 Passive Control Devices

- Do not require external power.
- Examples:
  - Base isolators
  - Tuned mass dampers
  - Viscous wall dampers

1.18.2 Active and Semi-Active Control

- **Active systems**: Use sensors and actuators to apply forces in real-time.
- **Semi-active**: Modify damping properties based on input signals.

1.18.3 Smart Materials in Vibration Control

- **Shape Memory Alloys (SMAs)**

- **Magneto-rheological (MR) dampers**
- Offer adaptive control during earthquakes, enhancing resilience.