Chapter 9: Fourier Integrals

Introduction

In practical applications, especially in Civil Engineering, many physical phenomena such as heat conduction, vibrations in structures, and signal transmission are best described using functions that may not be periodic. While Fourier Series is a powerful tool for representing periodic functions, it becomes inadequate when dealing with non-periodic functions. To handle such cases, **Fourier Integrals** are introduced.

A Fourier Integral allows us to express a non-periodic function as a continuous superposition of sines and cosines (or exponential functions), making it indispensable in solving engineering problems involving non-periodic boundary conditions and transient phenomena.

9.1 The Need for Fourier Integrals

Fourier Series represents a function $f(x)$ defined on a finite interval $[-L, L]$ as an infinite sum of sines and cosines:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

But for non-periodic functions or when the interval becomes infinitely large $(L \to \infty)$, the discrete nature of the Fourier coefficients becomes continuous, and the sum becomes an integral. This transition leads us to Fourier Integrals.

9.2 Derivation of the Fourier Integral

Let $f(x)$ be a piecewise continuous function on $(-\infty, \infty)$ that satisfies the Dirichlet conditions and is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$$

We begin with the Fourier series of $f(x)$ defined on $[-L, L]$:

$$f(x) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

Let $\omega_n = \frac{n\pi}{L}$. As $L \to \infty$, $\omega_n \to \omega$ becomes a continuous variable. The sum becomes an integral:

$$f(x) = \int_0^\infty \left[A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right]\, d\omega$$

Where,

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\cos(\omega t)\, dt, \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\sin(\omega t)\, dt$$

This is the **Fourier Integral Representation** of $f(x)$.

9.3 Fourier Integral Formula

Let $f(x)$ be a function such that:

- $f(x)$ is piecewise continuous on $(-\infty, \infty)$
- $f(x)$ is absolutely integrable over $(-\infty, \infty)$

Then,

$$f(x) = \int_0^\infty \left[A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right]\, d\omega$$

Where:

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\cos(\omega t)\, dt, \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\sin(\omega t)\, dt$$

Alternatively, combining cosine and sine terms, we can express the function using the **complex Fourier integral**:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \widehat{f}(\omega) e^{i\omega x}\, d\omega$$

Where:

$$\widehat{f}(\omega) = \int_{-\infty}^\infty f(t) e^{-i\omega t}\, dt$$

This is the **Fourier Transform** of $f(x)$.

9.4 Fourier Cosine and Sine Integrals

If $f(x)$ is even (i.e., $f(-x) = f(x)$), then its Fourier integral contains only cosine terms:

$$f(x) = \int_0^\infty A(\omega) \cos(\omega x) \, d\omega$$

If $f(x)$ is odd (i.e., $f(-x) = -f(x)$), then its Fourier integral contains only sine terms:

$$f(x) = \int_0^\infty B(\omega) \sin(\omega x) \, d\omega$$

This simplification is useful in boundary value problems involving symmetric domains.

9.5 Conditions for Fourier Integrability

For a function $f(x)$ to possess a valid Fourier Integral representation:

1. $f(x)$ must be piecewise continuous in every finite interval of $\mathbb{R}$.
2. $f(x)$ must be absolutely integrable over $(-\infty, \infty)$.
3. Discontinuities must be finite and of finite magnitude.

If these conditions are satisfied, then:

$$\lim_{\epsilon \to 0} f(x + \epsilon) + f(x - \epsilon) = 2f(x)$$

at all points of continuity.

9.6 Applications in Civil Engineering

Fourier Integrals are particularly important in Civil Engineering for solving:

- **Heat conduction problems** in infinite or semi-infinite rods
- **Vibration analysis** of continuous beams or plates
- **Dynamic analysis** of structures subject to non-periodic loading
- **Soil mechanics** for propagation of stress waves
- **Ground motion analysis** during earthquakes

For example, temperature distribution in a long concrete beam due to an instantaneous point source can be solved using the Fourier integral method.

9.7 Worked Examples

Example 1:

Evaluate the Fourier sine integral representation of the function:

$$f(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x \geq a \end{cases}$$

Solution:

Since $f(x)$ is odd over $(0, \infty)$, use the sine integral:

$$f(x) = \int_0^\infty B(\omega) \sin(\omega x)\, d\omega$$

Where:

$$B(\omega) = \frac{1}{\pi} \int_0^a 1 \cdot \sin(\omega t)\, dt = \frac{1}{\pi} \left[-\frac{\cos(\omega t)}{\omega} \right]_0^a = \frac{1}{\pi\omega} (1 - \cos(\omega a))$$

Thus,

$$f(x) = \int_0^\infty \frac{1 - \cos(\omega a)}{\pi\omega} \sin(\omega x)\, d\omega$$

Example 2:

Find the Fourier cosine integral of $f(x) = e^{-ax}$, $a > 0$, for $x > 0$.

Solution:

Use:

$$f(x) = \int_0^\infty A(\omega) \cos(\omega x)\, d\omega$$

$$A(\omega) = \frac{2}{\pi} \int_0^\infty e^{-at} \cos(\omega t)\, dt$$

Use known integral:

$$\int_0^\infty e^{-at} \cos(\omega t)\, dt = \frac{a}{a^2 + \omega^2}$$

Hence,

$$A(\omega) = \frac{2}{\pi} \cdot \frac{a}{a^2 + \omega^2}$$

Therefore,

$$f(x) = \int_0^\infty \frac{2a}{\pi(a^2 + \omega^2)} \cos(\omega x)\, d\omega$$

This verifies the cosine integral representation of $f(x) = e^{-ax}$.

9.8 Dirichlet's Integral

A useful result in Fourier Integrals is:

$$\int_0^\infty \frac{\sin \omega x}{\omega}\, d\omega = \begin{cases} \frac{\pi}{2}, & x > 0 \\ 0, & x = 0 \\ -\frac{\pi}{2}, & x < 0 \end{cases}$$

This integral frequently appears in solving boundary value problems using Fourier methods.

9.9 Complex Form of Fourier Integral

So far, we have discussed the Fourier integral in terms of sine and cosine functions. However, it is often more convenient and elegant to express it using complex exponentials.

Let $f(x)$ be an absolutely integrable function over $(-\infty, \infty)$. Then the **complex Fourier integral representation** of $f(x)$ is:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{i\omega x}\, d\omega$$

Where the Fourier transform $\widehat{f}(\omega)$ is defined as:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

The inverse Fourier transform is:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{i\omega x}\, d\omega$$

Advantages of the Complex Form:

- Unified treatment of both sine and cosine terms.
- Simplifies differential equation solutions in engineering.
- Well-suited for using Laplace and Fourier techniques together.

9.10 Properties of the Fourier Transform

Understanding the properties of the Fourier Transform helps in efficiently solving various problems. Let $f(x) \leftrightarrow \widehat{f}(\omega)$ denote the Fourier transform pair.

1. Linearity:

$$\mathcal{F}[af(x) + bg(x)] = a\widehat{f}(\omega) + b\widehat{g}(\omega)$$

2. Translation (Shift):

- In time domain:

$$\mathcal{F}[f(x-a)] = e^{-i\omega a}\widehat{f}(\omega)$$

- In frequency domain:

$$\mathcal{F}[e^{iax}f(x)] = \widehat{f}(\omega - a)$$

3. Scaling:

$$\mathcal{F}[f(ax)] = \frac{1}{|a|}\widehat{f}\left(\frac{\omega}{a}\right)$$

4. Differentiation:

If $f(x)$ is differentiable,

$$\mathcal{F}\left[\frac{d^n f(x)}{dx^n}\right] = (i\omega)^n \widehat{f}(\omega)$$

This is particularly useful for solving PDEs.

9.11 Fourier Integral in Engineering Problem Solving

Let's look at how the Fourier Integral is applied in a real engineering scenario.

Problem: Heat Diffusion in a Semi-Infinite Rod

A rod of infinite length initially at zero temperature receives an instantaneous unit heat source at $x = 0$ at $t = 0$. The governing equation is the **one-dimensional heat equation**:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

With initial condition:

$$u(x, 0) = \delta(x)$$

And boundary condition:

$$u(\pm\infty, t) = 0$$

Taking the Fourier Transform in x, we convert the PDE to an ODE:

$$\frac{\partial \widehat{u}(\omega, t)}{\partial t} = -\alpha^2 \omega^2 \widehat{u}(\omega, t)$$

Solving:

$$\widehat{u}(\omega, t) = e^{-\alpha^2 \omega^2 t}$$

Taking the inverse Fourier transform:

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha^2 \omega^2 t} e^{i\omega x} \, d\omega$$

Using the standard Gaussian integral identity:

$$u(x, t) = \frac{1}{\sqrt{4\pi\alpha^2 t}} e^{-\frac{x^2}{4\alpha^2 t}}$$

This is the **heat kernel**, a fundamental solution showing how heat diffuses through the rod — a crucial result for civil engineers analyzing thermal effects in structures.

9.12 Parseval's Theorem for Fourier Integrals

Parseval's identity relates the energy of a signal in time domain to that in frequency domain.

Let $f(x)$ and $g(x)$ be absolutely integrable functions. Then:

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}\,d\omega$$

When $f = g$, it becomes:

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(\omega)|^2\,d\omega$$

This has practical importance in **energy calculations, error estimation, and signal processing** in structural monitoring systems.

9.13 Comparison: Fourier Series vs Fourier Integral

Feature	Fourier Series	Fourier Integral
Applicable to	Periodic functions	Non-periodic functions
Domain	Finite interval	Infinite interval
Representation	Discrete sum of sines/cosines	Continuous integral
Coefficients	a_n, b_n	$A(\omega), B(\omega)$ or $\widehat{f}(\omega)$
Use in Engineering	Vibrations of bounded structures	Heat transfer, infinite domain analysis

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9.14 Common Integral Forms for Reference

To aid problem-solving, here are standard Fourier integral forms:

1. **Fourier Sine Transform of $f(x) = 1$, $0 < x < a$:**

$$\int_0^a \sin(\omega x)\,dx = \frac{1-\cos(\omega a)}{\omega}$$

2. **Fourier Cosine Transform of e^{-ax}:**

$$\int_0^\infty e^{-ax}\cos(\omega x)\,dx = \frac{a}{a^2+\omega^2}$$

3. **Fourier Sine Transform of e^{-ax}:**

$$\int_0^\infty e^{-ax}\sin(\omega x)\,dx = \frac{\omega}{a^2+\omega^2}$$

9.15 Exercises

1. Derive the Fourier cosine integral of $f(x) = xe^{-x}$ for $x > 0$.
2. Evaluate the Fourier sine integral of the step function:

$$f(x) = \begin{cases} 1, & 0 < x < L \\ 0, & x > L \end{cases}$$

3. Use the complex Fourier integral to represent $f(x) = \frac{1}{1+x^2}$.
4. Apply Fourier integral methods to solve the initial value problem for the one-dimensional wave equation.
5. Show that the function $f(x) = e^{-a|x|}$ has a Fourier transform and compute it.
