Hydraulic Engineering
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Lecture - 39
Pipe flow (Contd)

Welcome back student. We are into yet another lecture of pipe flow. Last time we finished talking about the entrance region.

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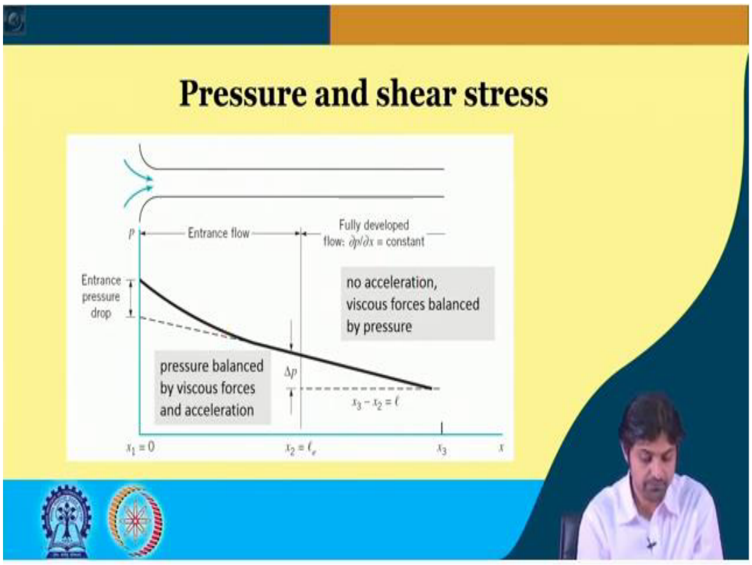

Now, we are going to see what the pressure and the shear stress distribution through a figure, you know, through a graph, in the both the entrance region and the fully developed flow region is. So, when the, as you see in this graph, as soon as the water enters the pipe there will be a pressure drop here and that is called the entrance pressure drop. This is le, as we said and this value can be calculated based on the Reynolds number.

If, the flow is laminar, it is 0.06 Re. Whereas, if it is turbulent, it is of the order of Re to the power 1/6. However, you see, after the flow has become fully developed, the pressure dp dx, you know, the pressure drop per unit length becomes constant. So, this has been obtained through experimental analysis. So, $\frac{\partial p}{\partial x}$ = constant. So, important information to grasp from this particular slide is that in the entrance there is an entrance pressure drop.

Whereas, when it becomes, the flow becomes fully developed $\frac{\partial p}{\partial x}$ is constant. Here, in the entrance flow what happens is, the pressure is balanced by the viscous forces and the acceleration in the entrance region. Whereas, in the fully developed flow there is no acceleration, no acceleration, therefore, the viscous forces are balanced only by the pressure drop.

So, one thing that distinguishes fully developed flow and the entrance flow is that the existence of acceleration in the entrance flow region compared to the fully developed flow region.

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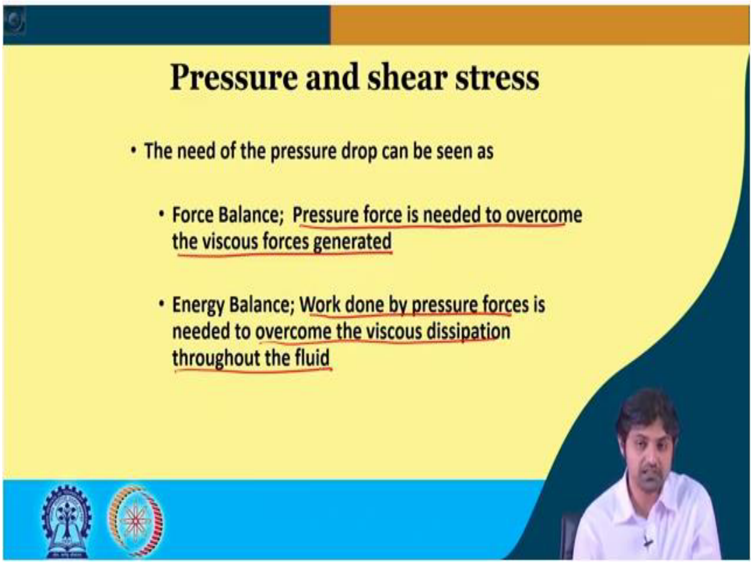

Now, the need of this pressure drop. The need of this pressure drop can be seen as, in terms of force balance, it can be said that the pressure force is needed to overcome the viscous forces generated. In terms, if we want to see why the pressure is needed to be dropped. So, pressure force is needed to overcome the viscous force generated.

Whereas, in terms of energy balance, we can say that the work which is done by the pressure forces is needed to overcome the viscous dissipation throughout the fluid. So, these are the 2 different ways of seeing the need of the pressure drop in the fully developed area and the same can also be applied for an entrance region. Just that the instead of only viscous forces, it will be viscous forces plus the acceleration.

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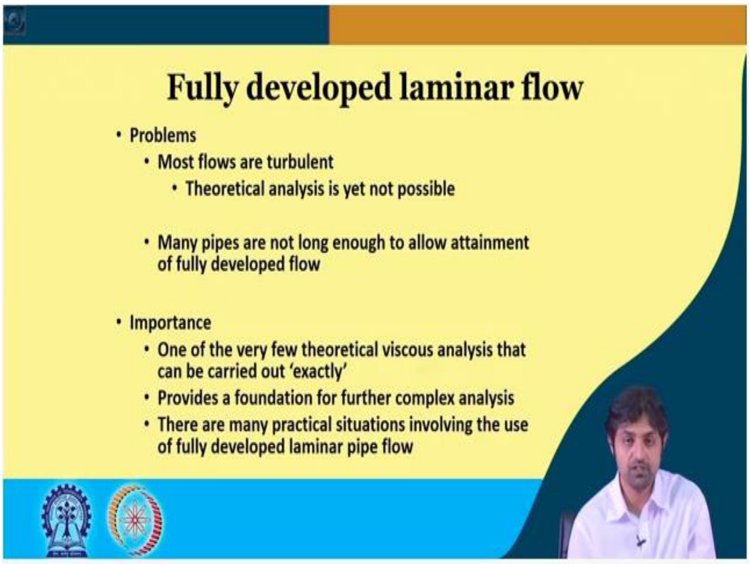

So, now, the problems with the fully developed laminar flow is that the most the, I mean, the basic problem is that in reality, most of the flows are actually turbulent. Therefore, the theoretical analysis is not yet possible. Second thing, most of the pipes that we see in our network are not long enough to allow the attainment of fully developed flow. Because if you see, it was le/D = 0.06 Re.

In case of, let us say Reynolds number of 4000, which is a pretty common, you know, this le and diameter of the pipe, let us say 1 meter, 0.06 into 4000. So, le becomes 240 meters. So, the entrance length region is 240 meters for a pipe of diameter 1 and Reynolds number of 4000. Even if the Reynolds number is 1000 then also it will require at least 60 meters length pipe. So, in most of the cases, what happens is these pipes are not long enough to allow attainment of fully developed flow because for that fully developed part, we can actually do the real analysis.

But that will occur only after the fully developed region has occurred, which in most cases does not, because the pipes are shorter in length. Now, but what is the importance of the fully developed laminar flow? There are certain problems related to it. But there are certain importances and advantages to it, as well. It is one of the very few theoretical viscous analysis that can be carried out exactly and that we will see how in our upcoming slides in lectures.

And therefore it also provides a foundation for further complex analysis. There are many practical situations which involves the use of fully develop laminar pipe flow. We will see those examples later.

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So, the equation for fully developed laminar flow in pipe can be derived using 3 approaches. What are these 3 approaches? One is from Newton's second law, which is applied directly. Second is from using the Navier-Stokes equation. The third one is from dimensional analysis. So what we are going to do? We are going to start the derivation of fully developed laminar flow in pipe, using Newton's second law now.

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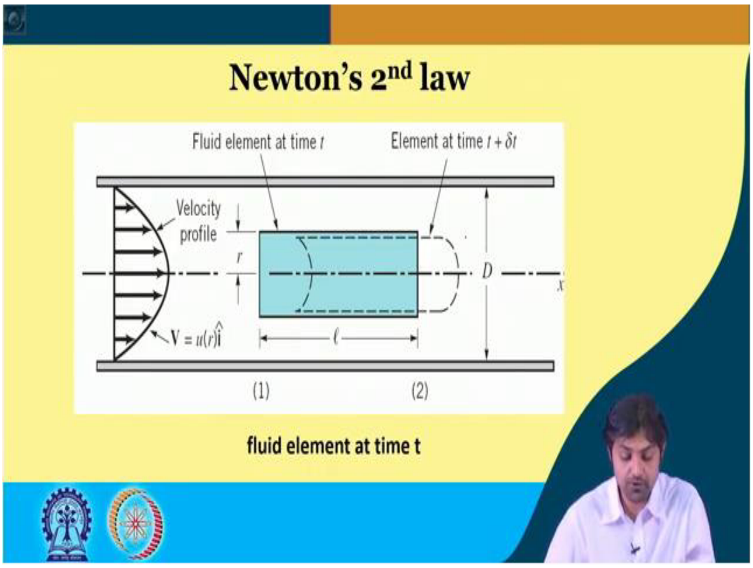

So, this is the snapshot of a fluid element at time t, this one here. And in the fully developed laminar flow this is the velocity profile, as we have seen in our laminar and turbulent flow analysis. And this velocity is only a function of radial distance r. from the pipe, this is the diameter D of the pipe, this is the x dimension and the fluid element is of length l, that we have considered.

At time t = t + delta t, so therefore, after at any time interval of delta t, this is the fluid element. This is section 1 and this is section 2. So, you have to keep this figure in mind.

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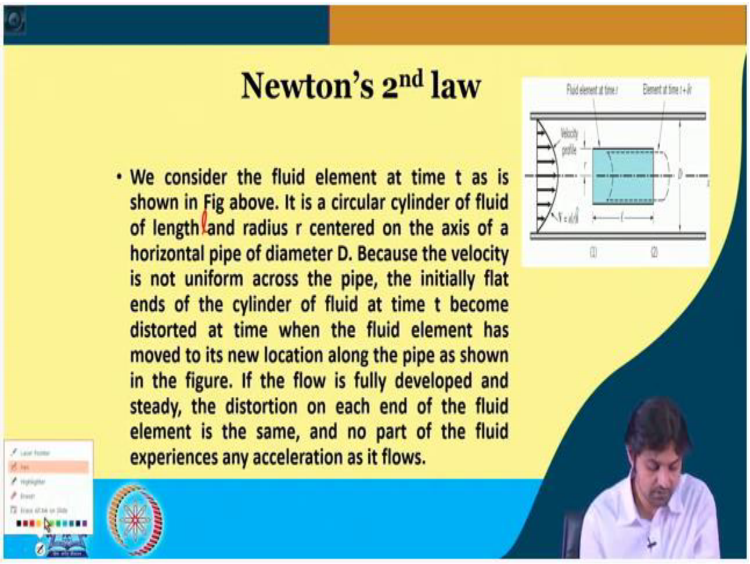

So, for your easiness I have already kept this figure here, so that you are able to follow the derivation. Now, I will repeat what I have said again. We have considered a fluid element at time t, as it is shown in the figure above. This element, so it is a circular cylinder of fluid of length l and radius r centered on the axis of a horizontal pipe of diameter D. Because the velocity is not uniform across the pipe, the initial flat end of the cylinder of fluid at time t becomes distorted at time when the fluid element has moved to its new location.

And the reason is, because across the radial distances, the velocity is not constant, it is a function of radial distance. If, the flow is fully developed and steady, steady means, that it is not a function of time, the distortion on each end of the fluid element is the same and no part of the fluid experiences any acceleration as it flows. So, this is the background for the derivation, containing important information that will be used during the derivation.

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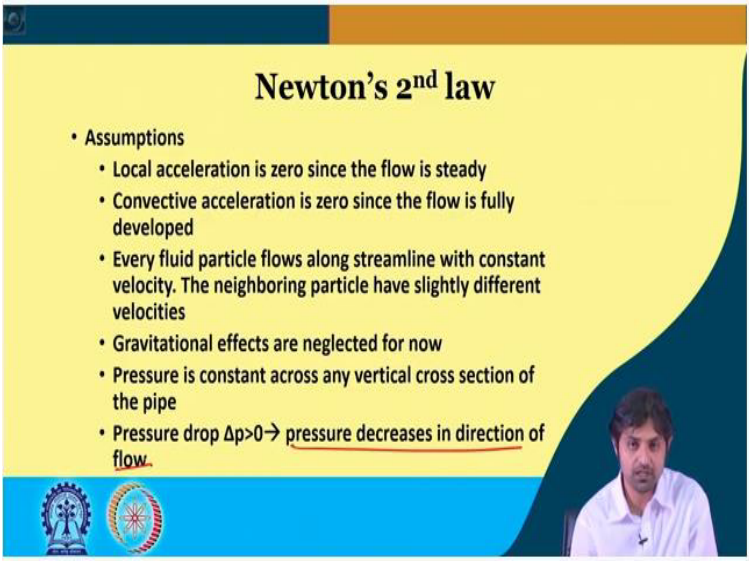

So, there are some assumptions. The assumptions is that the local acceleration is 0 since the flow is steady. We also assume that the convective acceleration is 0 since the flow is fully developed. Now, the every fluid particle flows along streamline with constant velocity. The neighboring particle have slightly different velocities, because the velocities are a function of radial distance r. For now, the gravitational effects will be neglected.

And the pressure is constant across any vertical cross section of the pipe. So, of course the pressure varies from this point, to this point, to this point, to this point, to this point, but across this cross section, the pressure if there is a pressure p here, it will be p here, it will be p here, p here. So, the pressure will is dependent, I mean, changes only with the x distance, not with the r distance.

Now, the pressure dropped delta p = 0 that means pressure decreases in the direction of flow. And that is very true, the water will move from a pressure from the region of having high pressure to that having low pressure.

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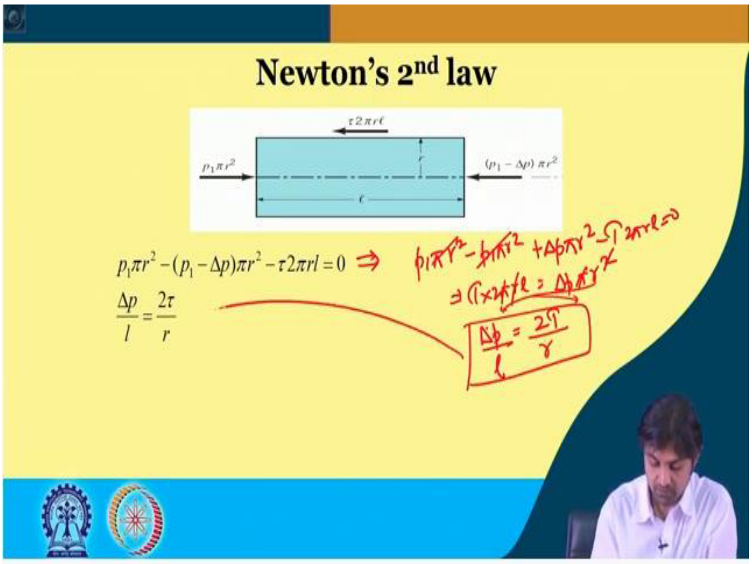

So, this is the figure for that element that we have drawn. So, the force acting from this side, if there is a pressure p1, on the left side at section 1 and this is section 2 and if we assume the delta p is the pressure drop, so p will be here, will be p2 is going to be, p1 - Δ p. And because of the viscous forces, if there is a shear stress acting called tau, then the force acting F1 will be pi 1 pressure into area.

Similarly, the another force will be acting here, F2 p2 into area and this is the shear stress force due to shear stress. So, the equation is going to be F1 - F2 - the tau into 2 pi rl because this shear stress will be acting on the entire cylindrical parameter. So, F1 is p1 pi r square. So, F1 is this, F2 is p1 – delta p into pi r square – t into 2 pi rl = 0. So, just going through, it will be $p_1\pi r^2 - (p_1 - \Delta p)\pi r^2 - \tau 2\pi r l = 0$, so expanding, so this, this gets cancelled.

So, we can write, $\tau 2\pi r l = \Delta p \pi r^2$. So, this r and this r gets cancelled, pi and pi gets cancelled. So, delta p can be written as, so delta p/l, so if l we bring this side, so delta p/l can be written as 2tau by, and this r we bring downside here. This is exactly same, as we have written here. So, we get, $\frac{\Delta p}{l} = \frac{2\tau}{r}$.

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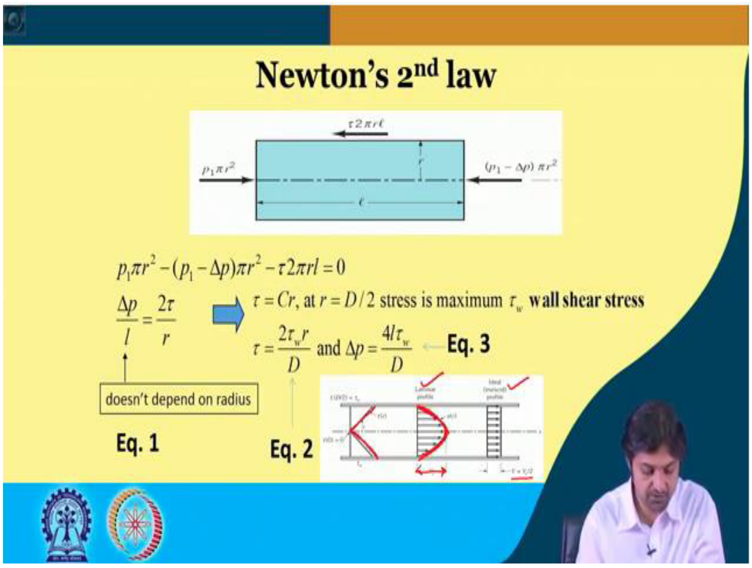

See, the pressure drop per unit length does not depend on, so this delta, sorry, so this delta p/l does not depend upon the radius. Because it is pressure drop per unit length and this we call as equation number 1. So, you see, we can say that delta p/l was constant. So, we can say tau is equal to, so we have got 2tau/r is equal to C1 or we can simply say tau = C into r. This is what we can get. This is C1, so but 2C is C1/2. So, tau we can write Cr.

So, we see, that at midpoint, at r = D/2, this tau takes CD/2 and that is the maximum. And that happens where? It happens at the wall. Therefore, this shear stress tau, at wall is called as tau w wall shear stress. So, if we put, r = D/2 in this equation, see. So, what do we get? So, tau = C into D/2. And this C is going to be 2 tau w/D. Therefore, tau can be written as, 2tau w r into D.

And when we have obtained tau, we can write, delta p because delta p/l = 2 tau/r or 2/r or we can write, D/2 here and tau is 2 tau w r. So, this r and r gets cancelled, so this becomes delta p/l = 4 tau w/D and delta p comes 4 tau w into l/D, this same equation. Now, you would have a question that, how did we get C? So, let us see, just going back a little bit, you might have a question, how this C becomes 2 tau w / D. How?

So, see, we have written, tau = Cr. So, at r = D/2, tau = tau w. So, we substitute this, in this equation here, we get, tau w = CD/2. So, C becomes 2 tau w / D and this is what has been substituted here. So, we get tau as a function of the wall shear stress, 2 tau w / D into r. So, now, we have obtained tau, in terms of wall shear stress. Therefore, using this particular equation, we have obtained delta p.

So, these are some of the important results to remember. This tau = 2 tau w r /D is equation number 2. As you can see, the tau is a function of r and this pressure drop, pressure drop per unit length is constant, as 4 tau w/D. So, this is the profile, you see. So, this is the laminar profile for viscous pipe flow and this is the ideal profile, you see. This is Vc. You remember, in our laminar and turbulent fluid flow, we derived this, that Vc was Vc/2, the average, the ideal inviscid profile.

Whereas, it goes from 0 to Vc and the shear stress variation is like this. This is the laminar profile in the pipe flow and the shear stress distribution is like this. This is something important to remember, as well.

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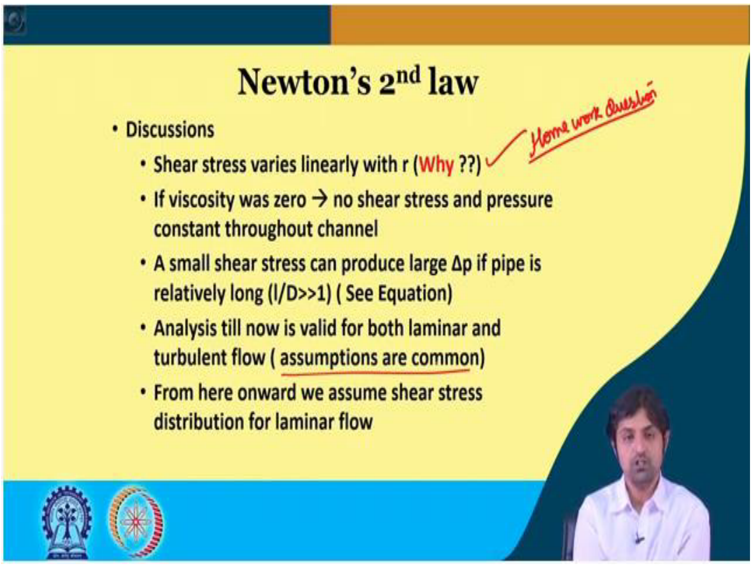

Now, the discussions. Shear stress varies linearly with r and why? If, the viscosity was 0 that there was no shear stress and pressure would have been constant throughout the channel. Therefore, a small shear stress can produce larger delta p if the pipe is relatively long. This you have to, you know, know why shear stress varies linearly with r. Of course, we have done the derivation but you try to find out.

So, you take it as a homework question. Secondly, if there was no viscosity, assume let us say, if it was, you know, 0 then there would be no shear stress and pressure would have been constant throughout the channel. That is true, because there would be nothing that would cause the pressure to work. Therefore, a small shear stress, you see this, you see, delta p is 4l tau w/D.

See, this equation and see our conclusion, that a small shear stress can produce large delta p, if the pipe is relatively long. So, if l/D is very, very much greater than 1 that then even a small shear stress will produce large delta p. So, until now, about the analysis we have done is valid for both laminar and turbulent flow. At no point in time, we have assumed that the flow yet is laminar. So, the assumptions for both laminar and turbulent flow is same.

So, all the equations, equation number 1, equation number 2 and equation number 3 are valid both for laminar and turbulent flow. So, from here onwards, we would assume the shear stress distribution for laminar flow. So, from now on in the next derivation, we would assume that the shear stress distribution is for the laminar flow.

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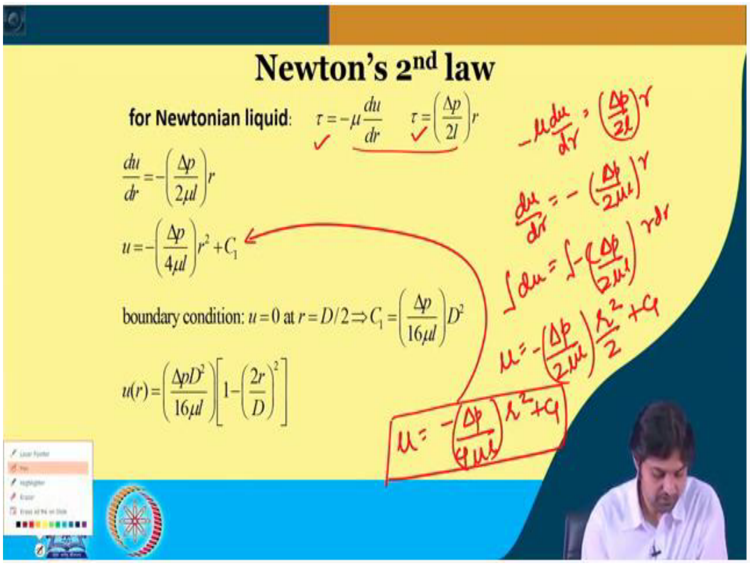

So, for Newtonian liquid, for laminar flow this is the shear stress distribution we have seen. Tau is is written as -Mu du dr, you remember this. This is what we have got, tau is equal to. How did we get this equation? We get this equation from our equation number 1. So, this is equation 1.

So, taking both together, we can see, see we equate these 2 and this becomes -Mu du dr = delta p/2 l into r. So, we equate this one and we equate this one.

So, du dr can be written as, minus of delta p/2 Mu l into r. So, if we take dr on this side, it will become du = - delta p/2 Mu l r dr. Therefore, we can simply write, integrate on both side, we get u = - delta p/2 Mu l with minus sign and r square/2 + C1. So, finally it can be written as, -delta p/4 Mu l r square + C1 and this is the same thing that we have written here. If, you can follow this.

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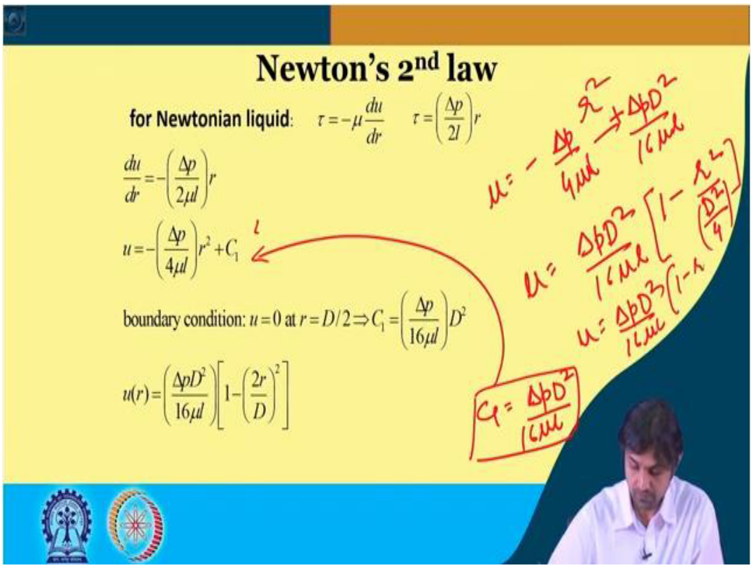

Now, importantly, there is 1 boundary condition at, so one of the boundary condition is, boundary condition is no slip condition. And where does it happen? It occurs at the wall. Or wall is where? At r = D/2, which means, u = 0 at r = D/2. And if we put this, in this equation, let us put it, we say 0 = -delta p/4 Mu l D square/4 + C1. So, what do we get? We get C1 as, delta p D square/16 Mu l. So, this is the value of C1 that we get.

And if we substitute this C1 even here, let us do that. So, I rub these. So, u will be minus delta p/4 Mu l into r square + C1 is delta p D square/16 Mu l. This r square can be written as, D square/4, therefore, if we take delta p D square/16 Mu l, as constant. Sorry, so this is not D, this is r square, so let me, so this is r square. So, u is, so let us take delta p D square/16 Mu l, as constant and then we are left with 1minus, so this will be 1 and this will be r square/D square/4 or exactly same. So this is what we get. So, we have got ur as a function of radial distance r.

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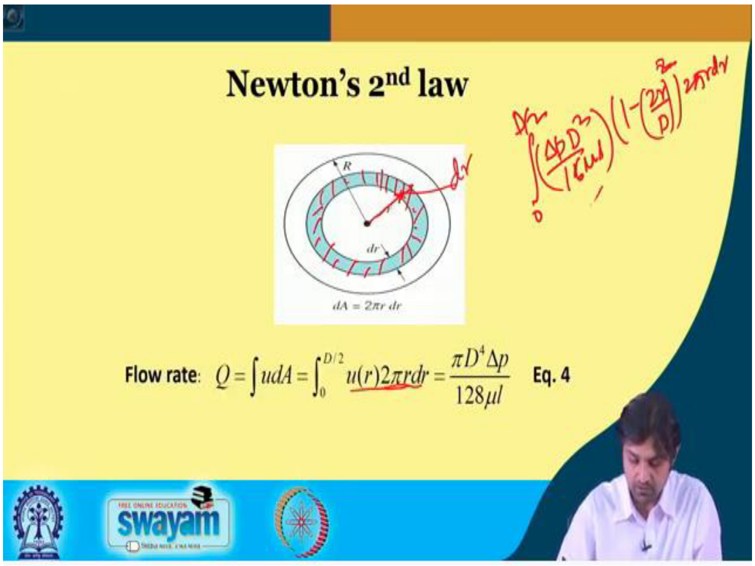

So, the flow rate is going to be, so now, we have determined u as a function of r and the flow rate across the entire pipe will be integral of 2 pi r dr, you see, the area. If, we start integrating at a distance, we take an element of thickness dr, at a distance r, we can write, Q will be u integral dA and we integrate from r = 0 to D/2 that the half, I mean, the entire radius, we can write, ur into 2 pi r dr.

So, integral 0 to D/2, u r was delta p D square/16 Mu l into 1 – 2r/D into 2 pi r dr. And on integration, we will come up with this equation

$$Q = \int u dA = \int_0^{D/2} u(r) 2\pi r dr = \frac{\pi D^4 \Delta p}{128 \mu l}$$

. This is the flow rate Q, in the entire pipe. This is termed as, equation number 4. And this particular flow rate Q is called the Poiseuille's law. So, we have used Newton's second law to derive the discharge rate through a pipe as a function of pressure gradient delta p and this is Poiseuille's law.

So, this is an important topic, I mean, important topic that this Poiseuille's law. So, one of the, you know, critical one to this laminar flow through pipes. So, I think, we will finish this lecture at this particular point. When we meet again in the next lecture, we see, how these laws can be modified if the gravity is also present and proceed further with our analysis. So, thank you so much for listening and attending today's class. I will see you in the next lecture.