Chapter 15: Fourier Integral to Laplace Transforms

15.1 Introduction

In the study of engineering mathematics, particularly for civil engineering applications, understanding how different integral transforms work is crucial. These transforms simplify complex differential equations and boundary value problems, converting them into algebraic forms that are easier to handle. Two of the most powerful tools in this regard are the **Fourier** and **Laplace transforms**.

This chapter discusses the **transition from Fourier Integrals to Laplace Transforms**, elaborating on how they relate, how they are used, and how each can be applied to solve real-world engineering problems such as beam deflection, heat conduction, and fluid flow.

15.2 Fourier Integral Theorem

Fourier Integral Theorem allows the representation of non-periodic functions as an integral (continuous sum) of sines and cosines.

15.2.1 Statement

Let $f(x)$ be a piecewise continuous function on every finite interval, absolutely integrable on the real line. Then,

$$f(x) = \lim_{M \to \infty} \int_{-M}^{M} \widehat{f}(\omega) e^{i\omega x} \, d\omega$$

where $\widehat{f}(\omega)$ is the Fourier transform of $f(x)$, defined as:

$$\widehat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

15.2.2 Fourier Integral Representation (Real Form)

If $f(x)$ is even:

$$f(x) = \int_0^{\infty} A(\omega) \cos(\omega x) \, d\omega$$

If $f(x)$ is odd:

$$f(x) = \int_0^\infty B(\omega) \sin(\omega x) \, d\omega$$

Where:

- $A(\omega) = \frac{1}{\pi} \int_0^\infty f(t) \cos(\omega t) \, dt$
- $B(\omega) = \frac{1}{\pi} \int_0^\infty f(t) \sin(\omega t) \, dt$

15.3 Fourier Cosine and Sine Transforms

15.3.1 Fourier Cosine Transform (FCT)

$$F_c(\omega) = \int_0^\infty f(x) \cos(\omega x) \, dx$$

Inverse:

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(\omega) \cos(\omega x) \, d\omega$$

15.3.2 Fourier Sine Transform (FST)

$$F_s(\omega) = \int_0^\infty f(x) \sin(\omega x) \, dx$$

Inverse:

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\omega) \sin(\omega x) \, d\omega$$

These transforms are particularly useful for solving partial differential equations (PDEs) in semi-infinite domains.

15.4 Limitations of Fourier Transforms

Although Fourier transforms are powerful for analyzing frequency components, they require functions to be integrable over the entire real line. In civil engineering applications, we often deal with causal systems defined only for $t \geq 0$. This is where **Laplace transforms** become highly valuable.

15.5 Transition to Laplace Transform

15.5.1 Motivation

Laplace transforms overcome the limitation of the Fourier transform by handling:

- Functions not absolutely integrable over $(-\infty, \infty)$
- Discontinuous and exponentially growing functions
- Initial-value problems in ordinary differential equations (ODEs)

15.5.2 Defining the Laplace Transform

Let $f(t)$ be defined for $t \geq 0$. The **Laplace transform** is:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Where s is a complex number: $s = \sigma + i\omega$.

The Laplace transform can be seen as a generalization of the Fourier transform by replacing $i\omega$ with a complex variable s.

15.6 Connection Between Fourier and Laplace Transforms

To bridge the two:

- Set $s = i\omega$ in the Laplace Transform.
- This makes the Laplace transform become a **bilateral Fourier transform** under certain convergence conditions.

15.6.1 Laplace Transform as a Modified Fourier Transform

Let:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-\sigma t} e^{-i\omega t} f(t)\, dt$$

This can be viewed as a Fourier transform of the function $e^{-\sigma t} f(t)$, provided the function is exponentially bounded.

Hence, the Laplace transform introduces a **damping factor** $e^{-\sigma t}$, which improves convergence.

15.7 Properties of Laplace Transforms

15.7.1 Linearity

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

15.7.2 First Shifting Theorem

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

15.7.3 Derivative Theorem

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

Useful for solving differential equations with initial conditions.

15.7.4 Integration Theorem

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{F(s)}{s}$$

15.8 Inverse Laplace Transform

Defined as:

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Using partial fraction decomposition and known transforms, the inverse is obtained.

15.9 Laplace Transform of Standard Functions

Function $f(t)$	Laplace Transform $F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$

15.10 Applications in Civil Engineering

15.10.1 Structural Vibrations

- Modeling free or forced vibrations of beams using differential equations.
- Laplace transforms simplify the equations and provide time-domain responses.

15.10.2 Heat Conduction Problems

- Fourier transforms help in solving 1D or 2D heat equations on infinite domains.
- Laplace transforms are used for transient heat conduction in semi-infinite media.

15.10.3 Groundwater Flow and Fluid Mechanics

- Laplace transforms solve unsteady flow equations governed by Darcy's law and continuity equations.

15.11 Comparison Table: Fourier vs Laplace

Feature	Fourier Transform	Laplace Transform
Domain	$(-\infty, \infty)$	$[0, \infty)$
Convergence	Requires function to be integrable	Exponentially bounded
Application	Frequency analysis	Time-domain ODE/PDE
Output	Function of ω (frequency)	Function of s (complex)

15.12 Laplace Transform of Piecewise and Discontinuous Functions

In civil engineering applications such as step loads, switching operations, and load removal, functions are often defined piecewise or are discontinuous. The Laplace transform can handle these efficiently using **Heaviside (unit step) functions**.

15.12.1 Unit Step Function $u(t-a)$

Defined as:

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

The Laplace Transform:

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

15.12.2 Transform of Shifted Functions

Let $f(t)$ be a function defined for $t \geq 0$, and consider $f(t-a)u(t-a)$, a shifted function starting at $t = a$. Then,

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

This is essential for modeling delayed responses in structures and loading systems.

15.13 Convolution Theorem for Laplace Transforms

The **convolution** of two functions $f(t)$ and $g(t)$ is defined as:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

Then,

$$\mathcal{L}\{(f * g)(t)\} = F(s) \cdot G(s)$$

This theorem is highly useful in systems analysis and inverse Laplace techniques, particularly when direct transformation is difficult.

15.14 Laplace Transform in Solving Differential Equations

Consider a second-order linear ODE with constant coefficients:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Apply Laplace Transform:

$$a[s^2Y(s) - sy_0 - y_1] + b[sY(s) - y_0] + cY(s) = F(s)$$

Solve algebraically for $Y(s)$, then find the inverse Laplace transform to obtain $y(t)$.

Example

Solve: $y'' + 3y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 0$

Step 1: Take Laplace on both sides:

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s+1} \Rightarrow Y(s)[s^2 + 3s + 2] = \frac{1}{s+1}$$

Step 2: Factor and solve:

$$Y(s) = \frac{1}{(s+1)(s+1)(s+2)}$$

Apply partial fractions, then take inverse Laplace to get $y(t)$.

15.15 Fourier Transform vs Laplace Transform in PDEs

15.15.1 Fourier Transform in PDEs

Used in problems with:

- Infinite or periodic domains
- Spatial analysis of signals or structures

Example: Heat equation $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ on $-\infty < x < \infty$

Apply Fourier transform on x-axis, solve ODE in t, then apply inverse transform.

15.15.2 Laplace Transform in PDEs

Used in problems with:

- Semi-infinite domains
- Initial conditions or transient analysis

Example: Same heat equation on $x \geq 0$, apply Laplace transform in t, solve spatial ODE in x, then invert.

15.16 Applications in Structural Dynamics

In real-world civil engineering, transient loads such as **earthquake forces, wind gusts, and vehicular impact** are modeled using time-dependent forcing functions. The governing equations are usually:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

Using Laplace transforms:

- Turn the differential equation into an algebraic equation in s
- Solve for $X(s)$
- Invert to get displacement $x(t)$

This gives insight into the **time history response** of structures.

15.17 Bromwich Integral and Laplace Inversion Formula

The inverse Laplace transform can also be expressed via a complex contour integral:

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\, ds$$

Where γ is a real number greater than the real parts of all singularities of $F(s)$. While not used in elementary courses, this provides a foundation for **complex analysis** and **residue theory**, which are advanced tools for engineers dealing with vibrations and stability.

15.18 Use of Laplace Transform in Finite Element Methods (FEM)

In numerical modeling (e.g., FEM for civil structures):

- Laplace transforms simplify the treatment of time-dependent boundary conditions.
- Time-stepping methods sometimes use Laplace-space solutions as initial conditions.
- Useful in transient heat transfer and stress-wave propagation.

15.19 Numerical Inversion of Laplace Transforms

In practice, analytical inversion is not always possible. Numerical methods are used:

- **Talbot’s method**
- **Durbin's method**
- **Zakian’s method**

These help civil engineers simulate **real-time system behavior**, particularly in **soil dynamics** and **hydrology**.
