Design and Analysis of Algorithms, Chennai Mathematical Institute
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Department of Computer Science and Engineering,

Week - 08
Module - 01
Lecture - 50
Linear Programming

The last week of this course, we looked at some advance topics. So let us begin with Linear Programming.

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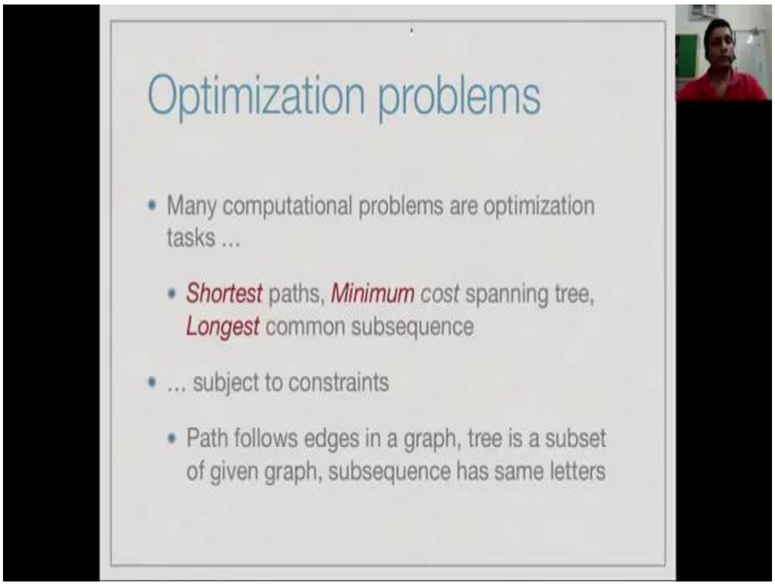

So, the times of problems we are looked at so far are largely optimization type of tasks. So, we are looking for shortest paths in a graph or we are trying to identify the minimum cost spanning tree or we are looking for the longest common sub sequence. So, we are trying to optimize and quantity the length of the path or the cost of the tree or the length of sub sequence, and then this come this optimization takes place subject to a constraint.

So, when we are looking for paths, the constraints are given by the graph itself, so the path must follow the given edges in the graph. So, as the graph changes the path that we identified will also change. Similarly, we are looking for spanning trees with respect to a given graph. So, in each graph we ask to first identify the different spanning trees and look for the smallest cost one among them. Similarly, with two sub sequences the actual letters in the sequence determine what is common between them.

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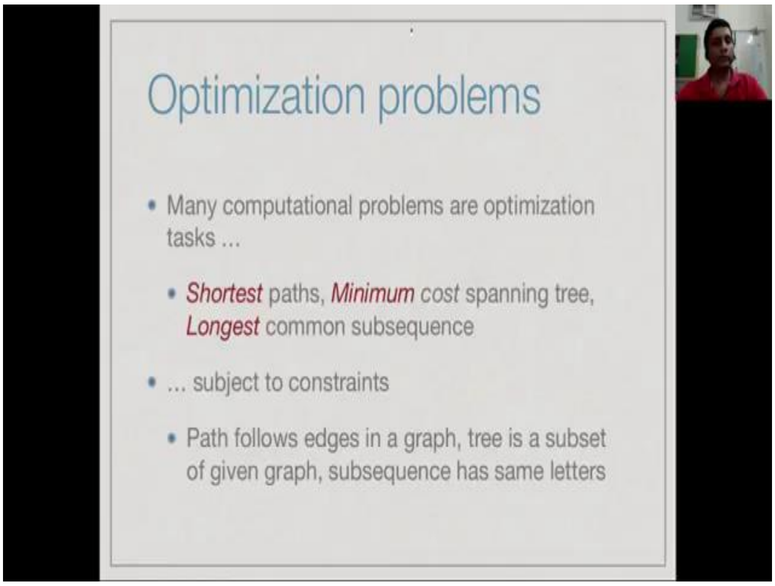

So, we can look at a node general formulation of such constraint optimization problems in the framework of what is called linear programming. So, in linear programming we are given some variables, some quantities that we want to calculate and then there are some linear functions that constraint these quantities. So, linear function remember of a variable x is something of the form a x plus b. So, it has no x square x cube term, it is all linear, so we have a x plus b.

So, in general then if we have multiple variables x y z, then we could have a constraint of the form a x plus b y plus something is less than equal to some constant or a x plus b y plus something is greater than equal to some constant. And then these are the constraints on the values that the variables can take and now our aim is to optimize some quantity, we want to minimize or maximize some cause or some weights or something like that and that quantity is expressed in terms of the variable by yet an another linear function.

So, we have some function which I will again write as a x plus b y, but it is a different set of coefficients of course, for each other constraints and for the objective function. So, we have a different linear function which tells us what it is we are trying to optimize.

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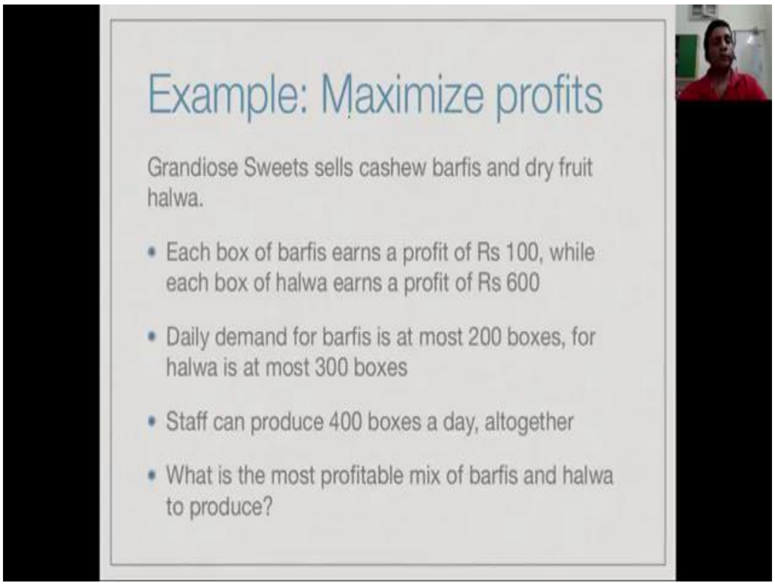

So, the best way to understand linear programming to begin with is to look at an example. So, suppose we are running a sweets shop called grandiose sweets and we tell sell two types of sweets, barfis and halwa. Now, we know that each box of barfis earns a profit of 100 rupees and each box of halwa earns a profit of 600 rupees. So obviously, make sense to make more halwa than barfis. Now, we also know that on a given day we cannot sell more than 200 barfi boxes, we cannot sell more than 300 halwa boxes and together the staff can only produce 400 boxes. So, now if you make 400 boxes, we can only sell 300 halwa, 200 barfis, so we need to make both. So, the question is given these constraints what combination of barfis and halwa should we make.

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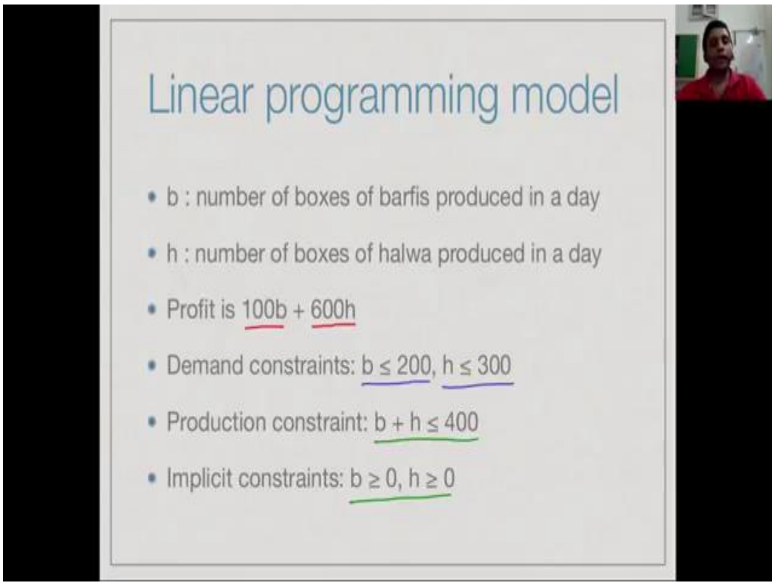

So, let us start by identifying the variables that we are trying to manipulate. So, I am talking about the daily production of these sweets, so we will use two variables b and h to denote the number of boxes of barfis and halwa that we produce in a day. Now, given this we know that our profit is 100 rupees per box of barfis and 600 rupees for box of halwa. So, it is totally 100 b plus 600 h, we also have some information about how much we can sell, so you know that we cannot sell more than 200 boxes of barfi a day.

So, b must be less than or equal to 200 and we cannot sell more than 300 boxes of halwa a day, so h must be less than or equal to 300. And finally, we have told that together our staff are only capable of producing 400 boxes all together, whether barfi or halwa any combination and implicitly of course, we can only make a non zero quantity of barfis, so we cannot make minus 7 boxes of barfis. So, we have these implicit constraints, we say that both b and h must be greater than zero. So, now we can put these all together into a set of inequalities and then an objective function and call it a linear program.

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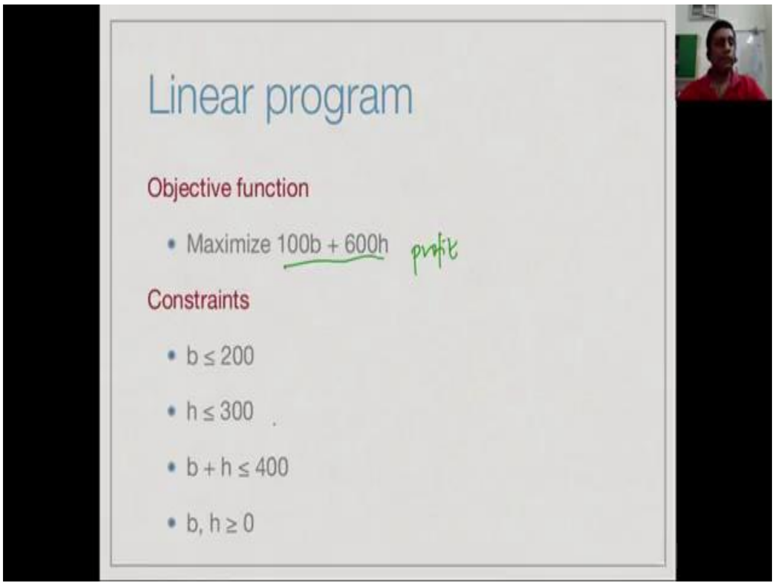

So, our objective is to maximize the profit, so this is the profit, so this is what we are going to maximize, the quantity 100 b plus 600 h and we cannot choose obviously, if b and h can vary, then we can make them as large as we want, but there are these constraints. So, b and h must lie between 0 and 200 and 0 and 300 respectively and together they can be no more than 400. So, these are the constraints we wrote in the previous page, now we are just writing them down, the system of equations are inequalities.

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So, we can visualize this quite easily in this case, because there are two quantities basically b and h that we want to compute and we are given a set of facts about them. So, we know that b lies between 0 and 200, so the range for b is anything in this region.

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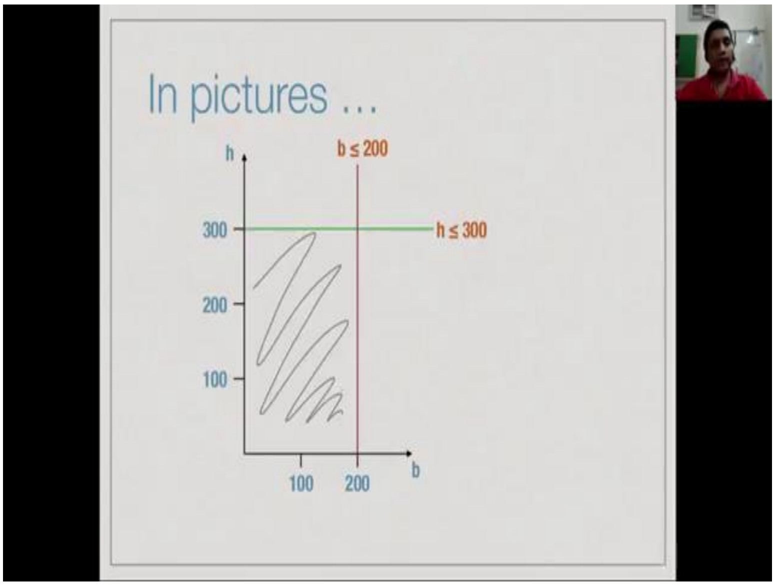

At the same time, we also know that edge lies between 0 and 300, so this constraints the space that we can look at to this right ((Refer Time: 05:30)).

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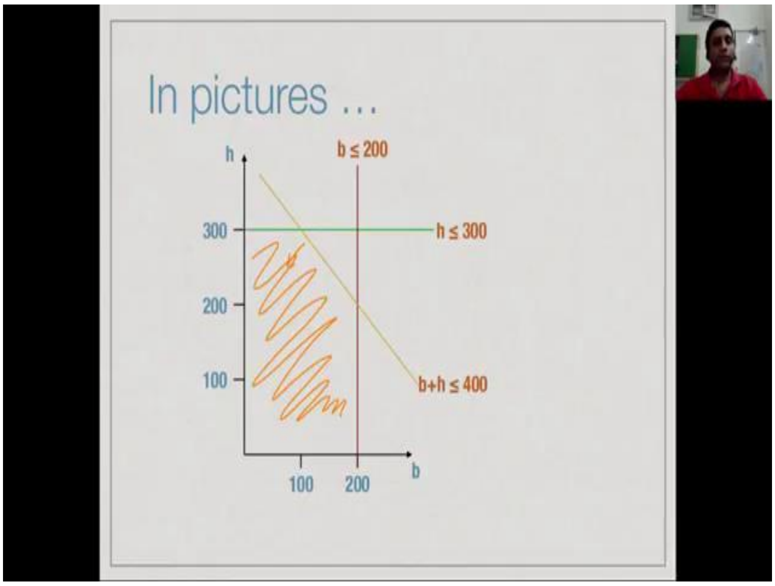

And finally, we have the other the constraint that together b plus h must be less than 400, so now that is all this side of this lines. So, we now have to be within this side of b c.

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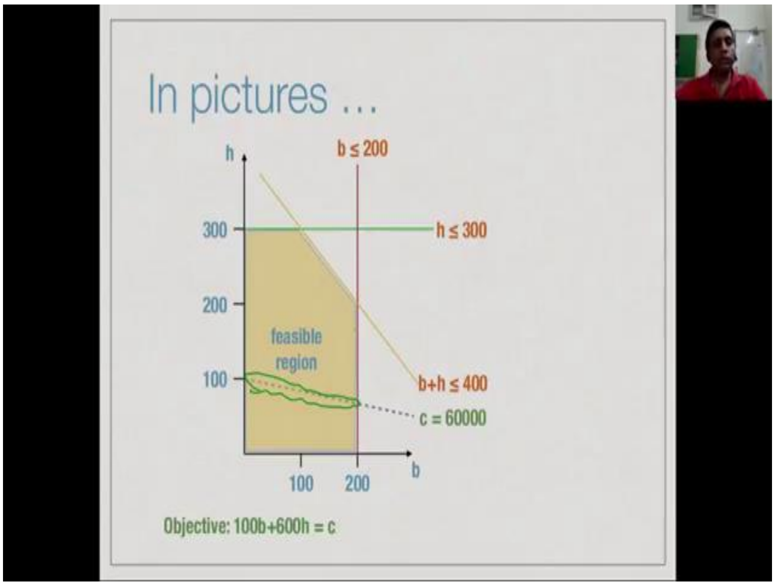

So, overall this gives us what we can call a feasible region, any point to in this region represents a combination of b value and h value which means all the constraints. Now, we have to introduce our profit function, so our objective is 100 b plus 600 h and depending on what values that b and h we choose, this will be equal to sum c. So, we can examine what this looks like for different values of c.

So, for example, if you set our profit to be 60,000 then any quantity along this line, any value of b and h which is inside the orange feasible region will give us 60,000. All these points give us 100 b plus 600 h equal to 60,000. Now, if I move this line across this region, the value changes.

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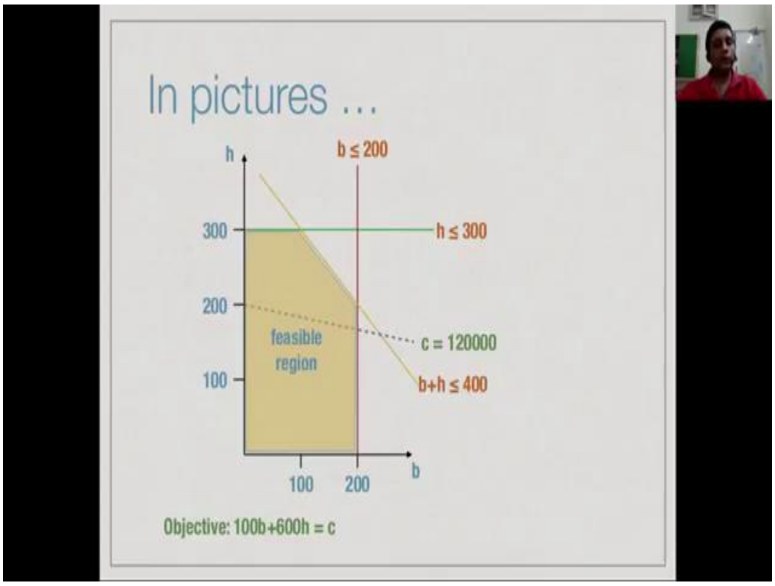

If I move it further or I can go from 60,000 to 1,20,000, 1 lakh 20,000.

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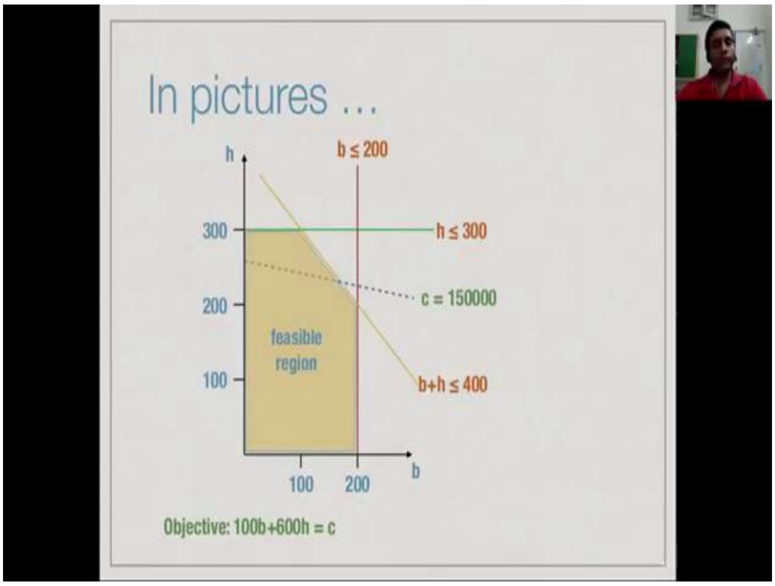

If I move it still further, I can get 1 lakh 50,000.

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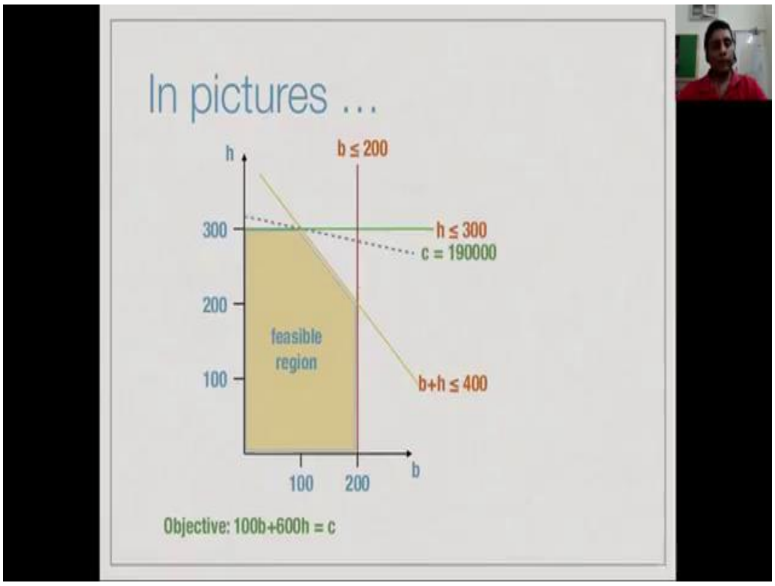

When that line actually leaves the region, at that corner vertex whether at pcm turns bends down, we find that we can get 1 lakh 90,000 and it turns out that this is actually going to be the optimum value. Now, this optimum value happens at some corner of this pcm and this is generally the case, one can argue that the optimal value will always occur at a vertex. Because, as we saw here the function we are trying to optimize the line and as we sweep across the feasible region, the value will keep either increasing or decreasing and so far where it leaves the region it will be some vertex.

Now, it could be that supposing this optimum line that actually been parallel to this, then both this vertex and this vertex would both have been optimal perhaps and all every point along this line. So, it is not saying that there are not optimal values away from vertices, but definitely there will be some vertex which has an optimum value. So, our goal is only to find an optimum solution, it is enough to look at the vertices.

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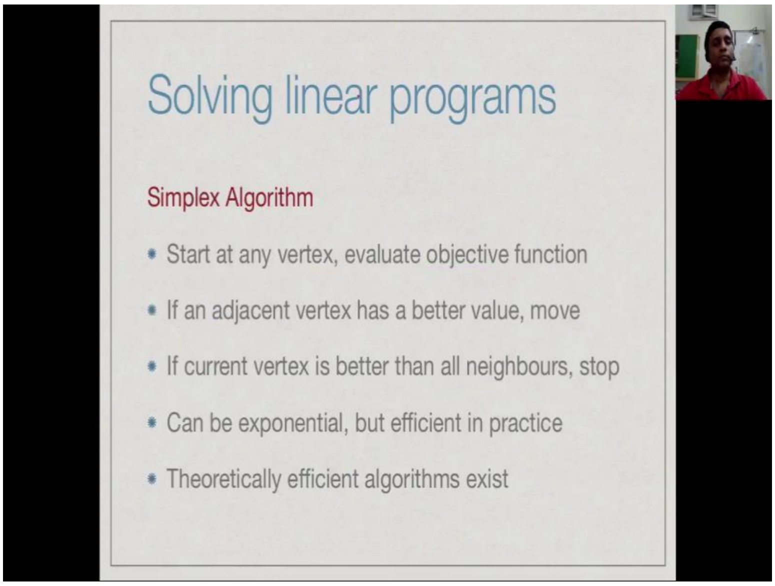

So, in fact, this is how the very famous simplest simplex algorithm works. So, what it does is, it constructs this feasible region which is bounded by a constraints and then it picks a vertex. So, we will not discuss it in detail, but you have to generalize this notion of a corner, in two dimensions it is quiet easy to imagine what a vertex is. But, in we have multiple constraints we have to optimize more than two dimensions, then you need to define vertex properly, but intuitively a vertex is a corner of this feasible region.

So, we start at any of this vertices, evaluate the objective function, then we look at the neighboring vertices. If any adjacent vertex, any neighbor has a better value we move to that neighbor, if none of our neighbors are better than us, you turns of the objective then we stop and announce that this vertex is the optimal. So, we will not going to a proof in this lecture that this is, if it is actually correct, but it is not hard to argue this, but there of course, a lot of details which need to be worked out to implement simplex correctly.

So, one of the problems which simplex is theoretically can be exponential, but actually in practice it turns out to be a very effective algorithm to solve linear programs. Now, there are theoretically efficient algorithms, so there is a polynomial time algorithm for linear programming, is there are so called the interior point methods and there is a method due to Narendra Karmarkar which many people may have heard of.

These are more complicated to describe and programs, simplex is probably easier to program, but what we need to know as far as we may concerned is that once you set up a set of constraints and an objective function using linear constraints, you can solve it efficiently.

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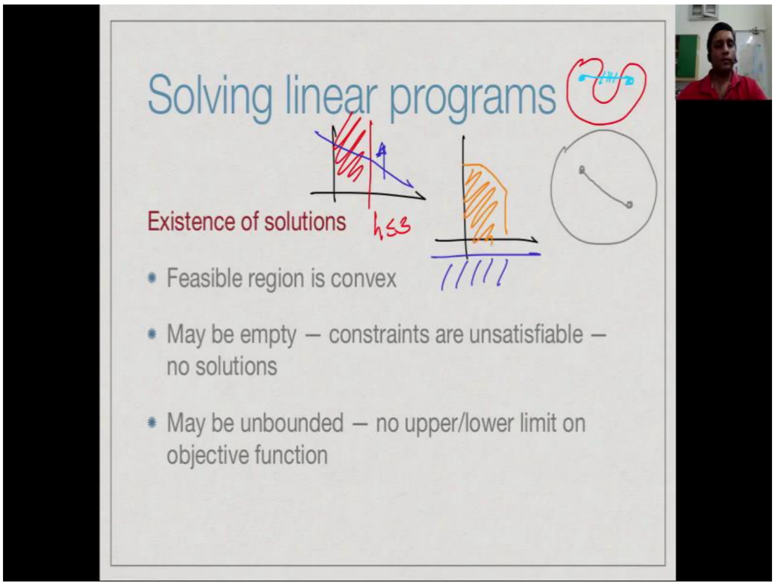

So, of course, you also across when a solution is exist. So, in our case we had a solution where we had a feasible region which looks like a trapezium like this. So, the first point is that the feasible region will already be convex, convex means that if you take any two points inside the region and you can connect them by a line, the entire line stays in the region. So, for instance if I have a shape which looks like this, then this is not convex, because I can pick two points inside and then connected by a line and this line in this portion leaves the region.

So, the feasible space will always be convex, but it could be empty. Supposing, we are had an extra constraint in an earlier thing which says that everything must be below this line, now when addition to the orange if I had this blue constraint, then there is no point that satisfies everything. So, the constraints are unsatisfiable, the feasible region become empty and there will be no solutions. The other possibility is that we do not constraint it enough, so we could just say that supposing we started only, we only say that say for example, h must be less than 300, then this has an unbounded region.

So, as I draw my blue line as I have done before and I keep moving it up, it says that profit keeps increasing, there is no upper point. So, there are these pathological cases where either the feasible region is empty or it is unbounded, where you will not find the solution. But, if it is bounded, then what the theory of linear programming tells us is that the objective function will lie, it is maximum or minimum will lie along on a vertex along the boundary of that bounded region and so it is sufficient to examine those and that is what simplex does.

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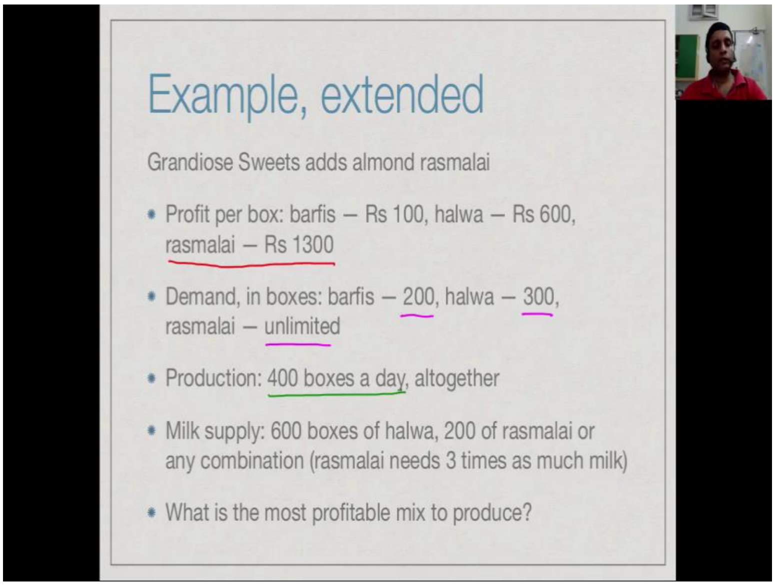

So, just to get a little bit more practice at this, let us extend the example. So, now, in addition to our barfi and halwa which have the same details that before a 100 rupee profit for barfis, 600 for halwa we added almond rasmalai which gives us a much higher profit of 1300. An almond rasmalai actually people are willing to buy in an unlimited quantity. So, the earlier demand for barfis and halwa is the same, 200 and 300, but almond rasmalai can be sold in unlimited amounts.

Unfortunately, we have not got any new staff, so we are still restricted to producing a total of 400 boxes and now we have an extra constraint which is in terms of what we can do with the milk that we get. So, with the milk that we get we can either makes 600 boxes of halwa in a day or to want a rasmalai or any combination of this. So, effectively a box of rasmalai requires three times as much as milk as a box of halwa. So, now, again given these constraints we want to find out what is the most optimum production schedule that we should compose on our sweet shop.

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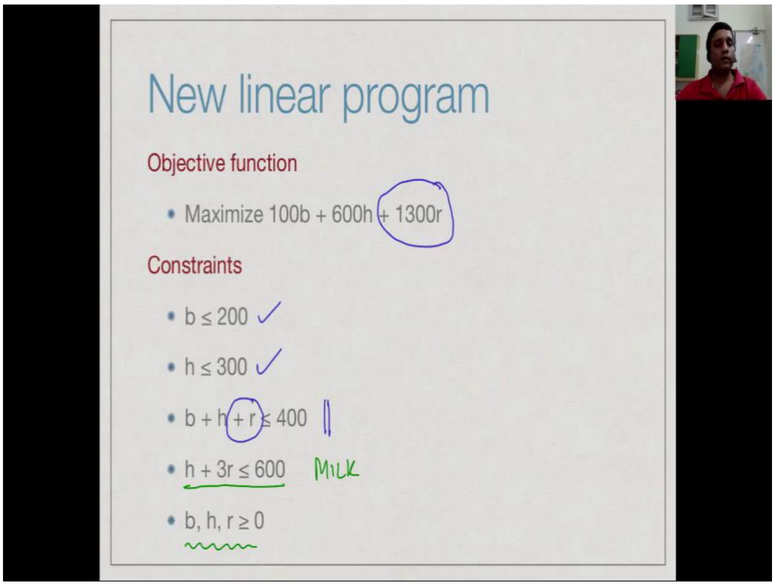

So, if we look at the equation now, the objective function now has an extra quantity which is the profit we get from rasmalai. So, we have an extra variable, we have b and h which are as before barfi and halwa and r for rasmalai, so we can get 100 rupees per box of barfi, 600 rupees per box of halwa and 1300 per box of rasmalai. The constraints on barfi and halwa demand of the same and there is no constraint for rasmalai, so we do not add anything there. The total production now includes the rasmalai, so all three together must be blue 400 and this expressives the milk constraint.

So, this is at one extreme, I can make 600 boxes of the halwa, at the other extreme I can make 200 boxes of rasmalai, because 200 times replace 600 rupees. But, I can make any combination and that would produces, requires the same amount of milk or less. So, h plus 3 r must be less than equal to 600, so this is the milk constraint. And finally, of course, all three of these quantities must be bigger than or equal to 0.

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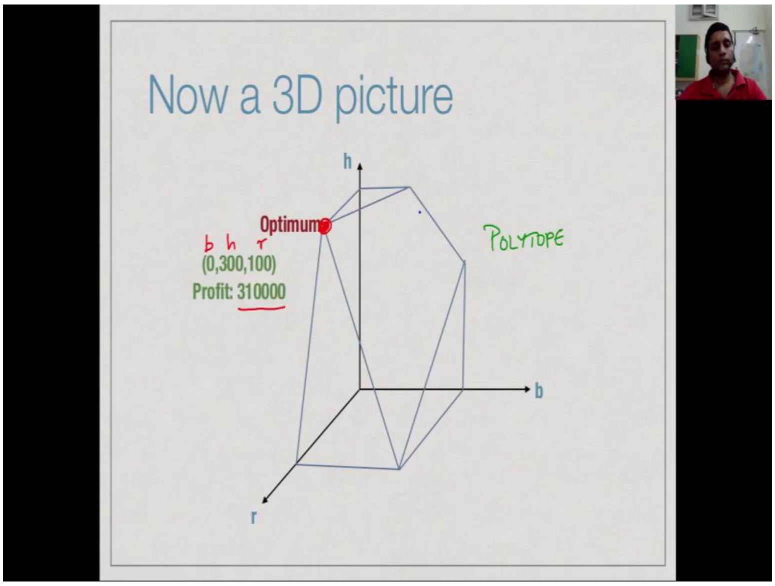

So, in this case if we draw this picture, now it becomes a three dimensional geometrical object, because we have three axis b, h and now r coming out in the vertical direction and the constraints now instead of the lines, they become these plans. So, the kind of become now instead of a polygon, we have what is called polytope, we have a three dimensional object. And again we will find that you get the optimum at some corner and the optimum happens to be in this case that you make zero barfis, you make a full amount 300 halwa and then you make the rest for rasmalai and then you get a profit of 3 lakhs 10,000.

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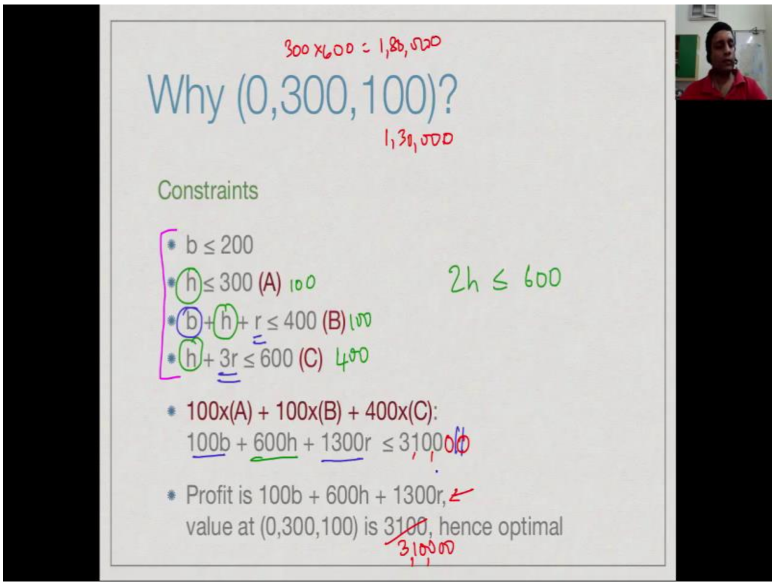

So, so we might ask is this really the case? So, is this really our best thing? So, remember that if I do this, I get 300 into 600, so I get 1 lakh 80,000 from here and you get 1 lakh

30,000 for here and thus how we get a 3 lakhs 10,000. So, is it how do I know that this answer is actually the best? So, here is an interesting observation that we can make. So, these are some of the constraints that we had before, we had just remove the one which says that the three values b, h and r are greater than zero, because that is not useful.

Now, in this supposing I take the three constraints labeled A, B and C. Now, in general if I take a constraint and I multiplied, so if I say 2 h is less than equal to 600. So, we know from simple algebra that this is same as h is less than equal to 300. So, in the same way I can multiply it is constraint by a constant, I come out, so combine them. I can take some combination of A plus some combination of B plus some combination of C, it tells us nothing new, we just combines it in a different format.

So, supposing I take 100 times this and I take 100 times B again and I take 400 times C and I add them, so I will get 100 h plus 100 h plus 400 h that is 600 h. Then I will get 100 b and finally, I will get 1200 plus 100, it is 1300 r. So, combining A, B and C with this combination or I multiply A by 100, B by 100, C by 400, I produce some new inequality. But, this new inequality tells us something interesting, because in fact what we have on the left hand side is our profit.

So, the profit is exactly 100 b plus 600 h, 1300 r, we are trying to maximize this and what our constraints tell us is that in the feasible region, this value can be no more than three ((Refer Time: 16:05)) should be, 3 lakhs 10,000. In other words, this is actually the optimum profit, because from our constraints we can derive the fact that we cannot get anything more than this, hence this must be an optimum value. So, in this particular example it seems that we can prove to ourselves that we have achieved an optimum by just doing something clever with that constraint.

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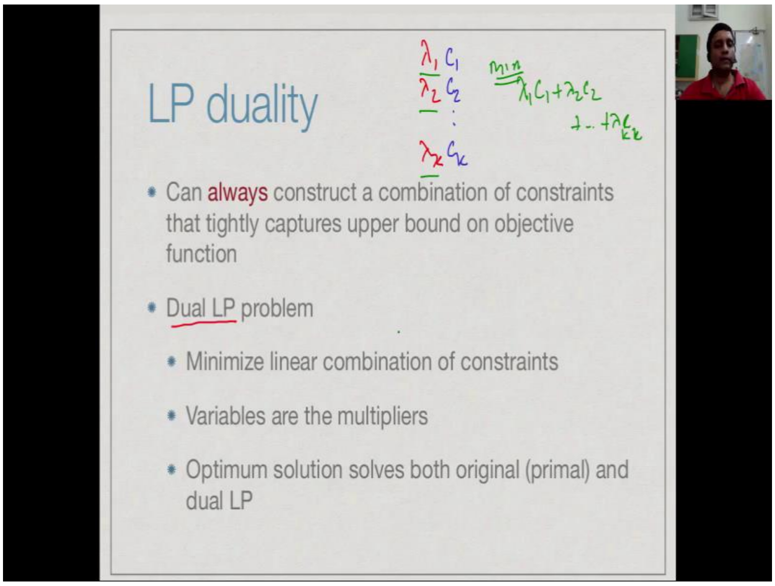

So, it turns out that fortunately this is not a coincidence, it turns out you can always construct such a combination. So, if I have my constraints C 1, C 2, so these are my constraint equations, I can always find some combination. So, I can take some lambda 1 C 1, lambda 2 C 2, I can add these up and then I can get from that some upper bound and that upper bound will actually tell me, whether not the solution I found is correct. So, this is called the dual of the LP and again we are not going to look into the theory of this, but this is useful to know that we can take the formulation that we start with for LP and then we can ask an another question which is what is the minimum value that we can get by combining the constraints with some linear multipliers.

So, we get new variables which are exactly this multiplier. What are the values of lambda 1 to lambda k which minimize the combination lambda 1 C 1 plus lambda 2 C 2 plus lambda k C k? So, this is my new question, I want to minimize this and then this becomes another LP problem and I solve that in a terms of that the dual and the original have a solution if and only if that solution is the optimum for both. So, this gives us the one way of justifying that I given linear programming problem has been solved correctly.