Chapter 8: Solution by Variation of Parameters

Introduction

In many engineering applications, especially in civil engineering, differential equations model real-world systems such as structural deflection, fluid flow, and heat conduction. Often, the equations encountered are **non-homogeneous linear differential equations**, where the method of *undetermined coefficients* may not be applicable due to the form of the non-homogeneous term. In such cases, the **method of variation of parameters** becomes a powerful and general technique to obtain a particular solution.

Unlike undetermined coefficients, which is limited to specific types of forcing functions (right-hand side), variation of parameters can handle a wider class of functions including logarithmic, exponential, and trigonometric terms or their combinations.

8.1 General Form of a Non-Homogeneous Second-Order Linear Differential Equation

A general second-order linear non-homogeneous differential equation is given by:

$$y'' + p(x)y' + q(x)y = g(x)$$

Where:

- y is the dependent variable (typically displacement, temperature, etc.)
- x is the independent variable (time, distance, etc.)
- $p(x)$, $q(x)$: Coefficient functions
- $g(x)$: Non-homogeneous term (external input or forcing function)

The **solution** to this equation is given by:

$$y(x) = y_h(x) + y_p(x)$$

Where:

- $y_h(x)$: General solution to the homogeneous equation $y''+p(x)y'+q(x)y = 0$
- $y_p(x)$: Particular solution to the non-homogeneous equation.

8.2 Principle of the Variation of Parameters

Suppose we already have the solution to the corresponding homogeneous equation:

$$y'' + p(x)y' + q(x)y = 0$$

Let the two linearly independent solutions of the homogeneous part be $y_1(x)$ and $y_2(x)$. Then the general solution of the homogeneous equation is:

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x)$$

To find a particular solution $y_p(x)$ to the non-homogeneous equation, we **assume**:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

Here, $u_1(x)$ and $u_2(x)$ are functions to be determined.

8.3 Derivation of the Variation of Parameters Formula

Differentiate $y_p(x)$:

$$y_{p'}(x) = u_{1'}(x)y_1(x) + u_1(x)y_{1'}(x) + u_{2'}(x)y_2(x) + u_2(x)y_{2'}(x)$$

To simplify the derivation, we impose a constraint:

$$u_{1'}(x)y_1(x) + u_{2'}(x)y_2(x) = 0$$

Then:

$$y_{p'}(x) = u_1(x)y_{1'}(x) + u_2(x)y_{2'}(x)$$

Differentiate again:

$$y_{p''}(x) = u_{1'}(x)y_{1'}(x) + u_1(x)y_{1''}(x) + u_{2'}(x)y_{2'}(x) + u_2(x)y_{2''}(x)$$

Now substitute $y_p, y_{p'}, y_{p''}$ into the original non-homogeneous equation:

$$y_{p''} + p(x)y_{p'} + q(x)y_p = g(x)$$

After substituting and simplifying using the homogeneous equation $y_{1''} + p(x)y_{1'} + q(x)y_1 = 0$ and similarly for y_2, we get:

$$u_{1'}(x)y_{1'}(x) + u_{2'}(x)y_{2'}(x) = g(x)$$

Now we have the system:

$$\begin{cases} u_{1'}(x)y_1(x) + u_{2'}(x)y_2(x) = 0 \\ u_{1'}(x)y_{1'}(x) + u_{2'}(x)y_{2'}(x) = g(x) \end{cases}$$

This is a **system of two linear equations in two unknowns**, $u_{1'}(x)$ and $u_{2'}(x)$. Solve using determinants (Cramer's rule).

Let $W(x)$ be the **Wronskian** of y_1 and y_2:

$$W(x) = y_1(x)y_{2'}(x) - y_{1'}(x)y_2(x)$$

Then:

$$u_{1'}(x) = -\frac{y_2(x)g(x)}{W(x)}, \quad u_{2'}(x) = \frac{y_1(x)g(x)}{W(x)}$$

Now integrate both:

$$u_1(x) = -\int \frac{y_2(x)g(x)}{W(x)}dx, \quad u_2(x) = \int \frac{y_1(x)g(x)}{W(x)}dx$$

Thus, the particular solution is:

$$y_p(x) = -y_1(x)\int \frac{y_2(x)g(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)g(x)}{W(x)}dx$$

8.4 Step-by-Step Procedure

1. **Solve the homogeneous equation** to find two linearly independent solutions $y_1(x), y_2(x)$.
2. **Compute the Wronskian** $W(x) = y_1y_{2'} - y_{1'}y_2$.
3. **Compute** $u_{1'}(x)$ **and** $u_{2'}(x)$ using:

$$u_{1'}(x) = -\frac{y_2(x)g(x)}{W(x)}, \quad u_{2'}(x) = \frac{y_1(x)g(x)}{W(x)}$$

4. **Integrate** to find $u_1(x), u_2(x)$.
5. **Construct the particular solution**:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

6. **Write the general solution** as:

$$y(x) = y_h(x) + y_p(x)$$

8.5 Example 1

Solve:

$$y'' - y = e^x$$

Step 1: Homogeneous equation:

$$y'' - y = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow r = \pm 1$$

So:

$$y_1(x) = e^x, \quad y_2(x) = e^{-x}$$

Step 2: Compute Wronskian:

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Step 3: Compute derivatives:

$$u_{1'} = -\frac{y_2 g}{W} = -\frac{e^{-x} \cdot e^x}{-2} = \frac{1}{2} \Rightarrow u_1 = \frac{x}{2}$$

$$u_{2'} = \frac{y_1 g}{W} = \frac{e^x \cdot e^x}{-2} = -\frac{e^{2x}}{2} \Rightarrow u_2 = -\frac{1}{4}e^{2x}$$

Step 4: Particular solution:

$$y_p = u_1 y_1 + u_2 y_2 = \frac{x}{2}e^x - \frac{1}{4}e^{2x}e^{-x} = \frac{x}{2}e^x - \frac{1}{4}e^x$$

So:

$$y(x) = C_1 e^x + C_2 e^{-x} + \left(\frac{x}{2} - \frac{1}{4}\right) e^x$$

8.6 Remarks on Usage in Engineering

- **Versatile**: This method can be applied to any function $g(x)$, unlike undetermined coefficients.
- **Computationally heavier**: Requires integration which might be complicated.
- **Useful in Civil Engineering** for modeling **forced vibrations**, **beam deflections under arbitrary loading**, **fluid flow**, and **non-constant boundary conditions**.

8.7 Advanced Example

Problem:

Solve the differential equation:

$$y'' + y = \tan x, \quad 0 < x < \frac{\pi}{2}$$

Step 1: Solve the homogeneous equation:

$$y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm i$$

So, the general solution to the homogeneous equation is:

$$y_h(x) = C_1 \cos x + C_2 \sin x$$

Hence, $y_1(x) = \cos x$, $y_2(x) = \sin x$

Step 2: Compute the Wronskian:

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Step 3: Compute $u_{1'}(x)$ and $u_{2'}(x)$:

$$u_{1'} = -\frac{y_2 g}{W} = -\sin x \cdot \tan x = -\frac{\sin^2 x}{\cos x}$$

$$u_{2'} = \frac{y_1 g}{W} = \cos x \cdot \tan x = \frac{\sin x}{1}$$

Now integrate:

$$u_1 = -\int \frac{\sin^2 x}{\cos x} dx = -\int \left(\frac{1 - \cos^2 x}{\cos x} \right) dx = -\int \sec x dx + \int \cos x dx$$

$$u_1 = -\ln|\sec x + \tan x| + \sin x$$

$$u_2 = \int \sin x dx = -\cos x$$

Step 4: Construct the particular solution:

$$y_p = u_1 y_1 + u_2 y_2 = (\sin x - \ln|\sec x + \tan x|)\cos x - \cos x \cdot \sin x$$

Simplify:

$$y_p = -\ln|\sec x + \tan x|\cos x$$

Final solution:

$$y(x) = C_1 \cos x + C_2 \sin x - \ln|\sec x + \tan x|\cos x$$

8.8 Common Mistakes and How to Avoid Them

1. **Wrong Wronskian sign or calculation:**
    - Always verify the Wronskian as incorrect W leads to wrong $u_{1'}, u_{2'}$.
    - Remember: $W(x) = y_1 y_{2'} - y_{1'} y_2$
2. **Forgetting constraints:**
    - When assuming $y_p = u_1 y_1 + u_2 y_2$, always apply the constraint:

$$u_{1'} y_1 + u_{2'} y_2 = 0$$

- This simplifies the second derivative.

3. **Difficult integrals:**
    - Not all integrals are elementary. Use substitution, parts, or computational tools when needed.
4. **Applying undetermined coefficients instead:**
    - If $g(x)$ is not a polynomial, exponential, sine, or cosine, *undetermined coefficients will not work.* Use variation of parameters.

8.9 Applications in Civil Engineering

1. Beam Deflection under Arbitrary Loads

In civil structures, especially in **beam theory**, the deflection $y(x)$ of a beam is governed by:

$$EI\frac{d^4y}{dx^4} = q(x)$$

Reducing this 4th-order equation step-by-step leads to a 2nd-order ODE of the form:

$$y'' + p(x)y' + q(x)y = g(x)$$

Here, $g(x)$ depends on the nature of the load $q(x)$, such as point loads or distributed loads.

- If $g(x)$ is a function like $\ln x$, $\tan x$, etc., **variation of parameters** must be used.
- Helps predict deflection of beams at any point along their span.

2. Vibration Analysis

In the presence of **external forces**, structural systems behave as:

$$my'' + cy' + ky = F(t)$$

- $F(t)$ being arbitrary, like $F(t) = \ln(t), t^2e^t$, etc.
- Variation of parameters gives a general method to find the **particular solution**, crucial in **dynamic response analysis**.

3. Hydraulic Engineering

Equations governing **unsteady open channel flows** or **groundwater flow** may lead to non-homogeneous ODEs with coefficients dependent on spatial variables.

8.10 Special Cases and Observations

1. **If $W(x) = 0$:**

   - The chosen functions y_1, y_2 are not linearly independent.
   - Cannot use variation of parameters unless new independent solutions are found.

2. **When the integral becomes too complex:**

- Use numerical methods or symbolic computation software (e.g., MATLAB, Mathematica).
- Sometimes, solutions may be expressed in terms of integrals (closed-form not possible).

3. **When g(x) is discontinuous:**
   - Break the domain into intervals where $g(x)$ is continuous and apply the method piecewise.
   - Use boundary/matching conditions at the discontinuity.

8.11 Graphical Interpretation

- The **homogeneous solution** represents the natural behavior of the system (e.g., natural oscillations of a beam).
- The **particular solution** represents the **forced response** due to external influences like loads, vibrations, or other input functions.
- In engineering practice, plotting the full solution $y(x) = y_h + y_p$ reveals insights about *resonance*, *maximum deflection*, and *instability zones*.
