Chapter 7: Solution by Undetermined Coefficients

Introduction

In the study of linear differential equations, especially those with constant coefficients, one of the key challenges is solving non-homogeneous equations. While the general solution of the homogeneous equation gives us the complementary function, the particular integral depends on the form of the non-homogeneous term. The method of **undetermined coefficients** is a powerful and straightforward technique for finding the particular integral when the non-homogeneous term is of a specific type — generally a polynomial, exponential, sine, or cosine function, or a combination of these.

This method is particularly useful for civil engineers because differential equations often arise in modeling structural systems, vibrations, beam deflections, and more, where external forces (non-homogeneous parts) can be expressed in such simple functional forms.

7.1 Overview of Linear Non-Homogeneous Differential Equations

A linear non-homogeneous differential equation of second order with constant coefficients can be written as:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where a, b, c are constants and $f(x) \neq 0$ is a known function (forcing function).

The general solution of such an equation is:

$$y(x) = y_c(x) + y_p(x)$$

- $y_c(x)$: Complementary function (solution of the homogeneous equation)
- $y_p(x)$: Particular solution (also called particular integral)

The method of **undetermined coefficients** is a technique to find $y_p(x)$, under certain conditions on $f(x)$.

7.2 Conditions for Applying the Method

This method works when the function $f(x)$ is of the following types:

- Polynomial (e.g., $f(x) = x^2 + 2x + 1$)
- Exponential (e.g., $f(x) = e^{kx}$)
- Sine or Cosine (e.g., $f(x) = \sin(ax)$, $\cos(bx)$)
- A product of the above (e.g., $f(x) = xe^{kx}$, $f(x) = x^2 \sin(x)$)

It is **not suitable** for functions like $\ln(x)$, $\tan(x)$, or when $f(x)$ is piecewise-defined or irregular.

7.3 Steps in the Method of Undetermined Coefficients

Step 1: Solve the Homogeneous Equation

$$ay'' + by' + cy = 0$$

Find the complementary function y_c by solving the **auxiliary equation**:

$$ar^2 + br + c = 0$$

The nature of roots (real and distinct, real and equal, or complex) determines the form of y_c.

Step 2: Guess the Form of the Particular Integral y_p

Assume a trial solution for y_p based on the form of $f(x)$, using undetermined coefficients.

Form of $f(x)$	Trial solution for y_p
$P_n(x)$ (polynomial of degree n)	$y_p = Ax^n + Bx^{n-1} + \ldots + C$
e^{kx}	$y_p = Ae^{kx}$
$\sin(ax)$, $\cos(ax)$	$y_p = A\cos(ax) + B\sin(ax)$
$x^m e^{kx}$	$y_p = (Ax^m + \ldots + C)e^{kx}$
$e^{kx}\sin(ax)$, $e^{kx}\cos(ax)$	$y_p = e^{kx}(A\cos(ax) + B\sin(ax))$

Step 3: Modify Trial Solution if Needed

If any term in the trial solution already appears in y_c, multiply the entire trial solution by x (or x^2, if needed) to eliminate duplication.

This adjustment is known as the **"annihilator approach"** or **"repetition rule."**

Step 4: Substitute and Determine Coefficients

Substitute the guessed y_p into the original non-homogeneous differential equation. Compare both sides and equate the coefficients of like terms to determine the unknown constants.

Step 5: Write the General Solution

$$y(x) = y_c(x) + y_p(x)$$

7.4 Illustrative Examples

Example 1: Exponential Forcing Function

Solve:

$$y'' - 3y' + 2y = e^x$$

Step 1: Solve the homogeneous part Auxiliary equation: $r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0 \Rightarrow r = 1, 2$

So,

$$y_c = C_1 e^x + C_2 e^{2x}$$

Step 2: Guess particular solution

Since e^x is already part of y_c, we guess:

$$y_p = Axe^x$$

Step 3: Substitute into original equation

Compute:

- $y_{p'} = Ae^x + Axe^x$
- $y_{p''} = Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x$

Substitute into LHS:

$$(2Ae^x + Axe^x) - 3(Ae^x + Axe^x) + 2Axe^x = e^x$$

Simplify:

$$2Ae^x + Axe^x - 3Ae^x - 3Axe^x + 2Axe^x = e^x \Rightarrow (-Ae^x) = e^x \Rightarrow A = -1$$

So,

$$y_p = -xe^x$$

Final Solution:

$$y(x) = C_1 e^x + C_2 e^{2x} - x e^x$$

Example 2: Polynomial Forcing Function

Solve:

$$y'' + y = x^2$$

Step 1: Homogeneous part: Auxiliary equation: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$$y_c = C_1 \cos(x) + C_2 \sin(x)$$

Step 2: Guess particular solution

Since RHS is x^2, try:

$$y_p = Ax^2 + Bx + C$$

Compute derivatives:

- $y_{p'} = 2Ax + B$
- $y_{p''} = 2A$

Substitute:

$$2A + (Ax^2 + Bx + C) = x^2 \Rightarrow Ax^2 + Bx + (2A + C) = x^2$$

Comparing both sides:

- $A = 1$
- $B = 0$
- $2A + C = 0 \Rightarrow 2 + C = 0 \Rightarrow C = -2$

So,

$$y_p = x^2 - 2$$

Final Solution:

$$y(x) = C_1 \cos(x) + C_2 \sin(x) + x^2 - 2$$

Example 3: Trigonometric Forcing Function

Solve:

$$y'' + 4y = \cos(2x)$$

Step 1: Homogeneous part: Auxiliary equation: $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$$y_c = C_1 \cos(2x) + C_2 \sin(2x)$$

Since $\cos(2x)$ is part of the complementary function, try:

$$y_p = x(A\cos(2x) + B\sin(2x))$$

Calculate derivatives and substitute into the equation, then compare coefficients to find A, B.

[**Detailed calculation omitted for brevity, should be worked out manually.**]

7.5 Summary Table of Trial Solutions

$f(x)$	Trial y_p	If duplication with $y_c \rightarrow$ Multiply by
x^n	Polynomial of same degree	x, x^2, $\ldots$ as needed
e^{ax}	Ae^{ax}	x, x^2, $\ldots$
$\sin(bx)$, $\cos(bx)$	$A\cos(bx) + B\sin(bx)$	x, x^2, $\ldots$
$x^n e^{ax}$	Polynomial $\times$ e^{ax}	Multiply entire guess by x
$e^{ax}\cos(bx)$, $\sin(bx)$	$e^{ax}(A\cos(bx) + B\sin(bx))$	x, x^2, $\ldots$

7.6 Theoretical Justification of the Method

The method of undetermined coefficients relies on the **superposition principle** and the fact that derivatives of certain function types (polynomials, exponentials, sine and cosine) reproduce the same function types.

Why it works:

- For linear differential equations with constant coefficients, if the non-homogeneous term $f(x)$ and its derivatives belong to a **closed class** of functions (i.e., differentiating does not produce new functional forms), then it is possible to guess a general form for y_p.
- The undetermined coefficients are then found by enforcing the equation structure, ensuring the guess satisfies the differential equation.

> **Note:** The method is not applicable when $f(x)$ is not from this class — for such cases, the **Variation of Parameters** method must be used (covered in Chapter 8).

7.7 Special Cases and Modifications

Case 1: Repeated Roots and Duplication

If the trial solution overlaps with any term of the complementary function, multiply by x^m, where m is the smallest integer such that there is **no duplication**.

> **Example:** If $f(x) = e^{2x}$, and the complementary function y_c includes e^{2x}, then instead of guessing Ae^{2x}, we guess Axe^{2x}. If e^{2x} appears **twice** in y_c (as in repeated roots), guess Ax^2e^{2x}.

Case 2: Non-Standard Right-Hand Side

If the non-homogeneous term involves:

- A combination like $x^2e^x \sin x$
- A rational function like $\frac{1}{x+1}$

The method **fails**. In such cases, use either **variation of parameters** or **Laplace transforms**.

7.8 Common Mistakes to Avoid

1. **Forgetting duplication with y_c:**
    - Always check the complementary function before deciding the trial solution.
2. **Incomplete form of the trial function:**
    - For polynomials of degree n, include **all** terms from x^n down to constant.
3. **Arithmetic and substitution errors:**

- During substitution, carefully compute derivatives and simplify terms precisely.

4. **Ignoring the domain of validity:**
   - Make sure the solution is valid for the domain specified in the problem, especially for initial/boundary value problems.

7.9 Applications in Civil Engineering

The method of undetermined coefficients is not just a theoretical tool. In civil engineering, it finds direct application in the following areas:

1. Structural Analysis

- When analyzing **beam deflections**, the governing differential equation involves external loads that are often modeled as polynomials or sinusoids. The method helps find the displacement function of the beam.

2. Mechanical Vibrations

- Damped or forced vibrations of structures like bridges, buildings, or machinery parts can be described by second-order linear differential equations. The forcing function (external periodic load) may be sinusoidal — a perfect candidate for this method.

3. Fluid Mechanics

- In modeling laminar flow between plates or through pipes, differential equations arise where pressure gradients or boundary influences are modeled as known functions.

4. Heat Transfer

- The steady-state heat distribution in a slab with internal heat generation can often lead to non-homogeneous linear ODEs solvable using this method.

7.10 Practice Problems

Problem 1:

Solve $y'' + 5y' + 6y = 3e^{-x}$

Problem 2:

Solve $y'' - 4y = x^2$

Problem 3:

Solve $y'' + y = x\sin(x)$

Problem 4:

Solve $y'' + 2y' + y = e^{-x}$

> *Hint: The complementary function here will include e^{-x}, so use modification.*

7.11 Advanced Extension: Higher-Order Equations

The method extends naturally to **higher-order linear ODEs** of the form:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y = f(x)$$

If $f(x)$ fits within the suitable function class, the trial solution follows the same logic:

- Match form of $f(x)$
- Include all terms needed
- Modify with powers of x if duplication occurs

  For example: For $y''' - 3y'' + 3y' - y = x^2$, the process is nearly the same, but you differentiate the trial function **three times**.
