Chapter 20: Rectangular Membrane, Use of Double Fourier Series

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Introduction

In structural and civil engineering, understanding how membranes (like vibrating plates, stretched rectangular sheets, etc.) behave under various conditions is crucial for design and analysis. A **rectangular membrane** is a two-dimensional object that can oscillate or vibrate when disturbed. Mathematically, its motion is governed by the **two-dimensional wave equation**, and one of the most powerful methods for solving such partial differential equations over rectangular domains is the **double Fourier series**.

This chapter focuses on formulating and solving problems involving the vibration of a **rectangular membrane** with fixed boundaries, using the method of **separation of variables** and **double Fourier series expansion**.

20.1 The Two-Dimensional Wave Equation

The transverse vibration $u(x, y, t)$ of a rectangular membrane stretched tightly and fixed at the boundary is governed by the **two-dimensional wave equation**:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

where:

- $u(x, y, t)$: displacement of the membrane at point (x, y) and time t,
- c: wave speed, depending on the tension and mass density of the membrane.

The rectangular membrane has dimensions $0 < x < a$, $0 < y < b$, and its edges are held fixed, leading to **boundary conditions**:

$$u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0$$

Additionally, the **initial conditions** are:

$$u(x, y, 0) = f(x, y), \qquad \left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x, y)$$

where $f(x, y)$ is the initial shape and $g(x, y)$ is the initial velocity distribution.

20.2 Solution by Separation of Variables

We assume a solution of the form:

$$u(x, y, t) = X(x)Y(y)T(t)$$

Substituting into the wave equation:

$$XY\frac{d^2T}{dt^2} = c^2T\left(Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2}\right)$$

Dividing both sides by XYT:

$$\frac{1}{T}\frac{d^2T}{dt^2} = c^2\left(\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2}\right)$$

Since the left side depends only on t and the right side only on x and y, both must equal a constant, say $-\lambda$. We then separate again for spatial parts:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = -\frac{1}{c^2T}\frac{d^2T}{dt^2} = -\lambda$$

Further separation gives:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2, \quad \text{so that} \quad \lambda = \alpha^2 + \beta^2$$

Then the time equation becomes:

$$\frac{d^2T}{dt^2} + c^2(\alpha^2 + \beta^2)T = 0$$

20.3 Solving the Spatial Equations

(i) Solving for $X(x)$:

$$\frac{d^2X}{dx^2} + \alpha^2X = 0$$

with boundary conditions: $X(0) = X(a) = 0$

Solution:

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \quad \alpha_n = \frac{n\pi}{a}, \quad n = 1, 2, 3, \ldots$$

(ii) Solving for $Y(y)$:

$$\frac{d^2Y}{dy^2} + \beta^2 Y = 0$$

with boundary conditions: $Y(0) = Y(b) = 0$

Solution:

$$Y_m(y) = \sin\left(\frac{m\pi y}{b}\right), \quad \beta_m = \frac{m\pi}{b}, \quad m = 1, 2, 3, \ldots$$

20.4 Solving the Time Equation

$$\frac{d^2T}{dt^2} + c^2\left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right)T = 0$$

Let:

$$\omega_{mn} = c\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

Then:

$$T_{mn}(t) = A_{mn}\cos(\omega_{mn}t) + B_{mn}\sin(\omega_{mn}t)$$

20.5 General Solution

Combining all parts, the full solution is:

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[A_{mn}\cos(\omega_{mn}t) + B_{mn}\sin(\omega_{mn}t)\right]\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

This is a **double Fourier series** expansion in terms of eigenfunctions in x and y.

20.6 Determining Coefficients A_{mn} and B_{mn}

Using initial conditions:

From $u(x, y, 0) = f(x, y)$**:**

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)$$

Apply **double Fourier sine series** to find:

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) dydx$$

From $\frac{\partial u}{\partial t}(x, y, 0) = g(x, y)$**:**

$$g(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \omega_{mn} B_{mn} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)$$

Then:

$$B_{mn} = \frac{4}{ab\,\omega_{mn}} \int_0^a \int_0^b g(x, y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) dydx$$

20.7 Modes of Vibration

Each pair (m, n) corresponds to a distinct **mode of vibration** with frequency ω_{mn}. The **fundamental mode** is for $m = n = 1$, and higher modes correspond to more complex patterns of vibration.

These modes are crucial in civil engineering, especially in the analysis of plates, slabs, and other planar structures, where resonance and dynamic load effects must be considered in design.

20.8 Applications in Civil Engineering

- **Vibrations of bridge decks, slabs, and floors**
- **Dynamic response analysis** of rectangular structural elements
- **Seismic analysis** of 2D surface structures
- **Sound and vibration insulation modeling**

In all such applications, double Fourier series methods provide exact or approximate analytical insight into the behavior of structural systems.

20.9 Orthogonality of Sine Functions

An essential property used in deriving Fourier coefficients is the **orthogonality** of sine functions. For integers $m, n \in \mathbb{N}$, we have:

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx = \begin{cases} 0, & m \neq n \\ \frac{a}{2}, & m = n \end{cases}$$

A similar identity holds for integration over y:

$$\int_0^b \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right) dy = \begin{cases} 0, & m \neq n \\ \frac{b}{2}, & m = n \end{cases}$$

These orthogonality properties ensure that each term in the double Fourier series is independent and allow for exact extraction of Fourier coefficients A_{mn} and B_{mn}.

20.10 Eigenvalues and Eigenfunctions

In the process of solving the boundary value problem using separation of variables, the values $\alpha_n = \frac{n\pi}{a}$, $\beta_m = \frac{m\pi}{b}$ serve as **eigenvalues**, and the corresponding sine functions:

$$\phi_n(x) = \sin\left(\frac{n\pi x}{a}\right), \quad \psi_m(y) = \sin\left(\frac{m\pi y}{b}\right)$$

are **eigenfunctions**.

Each mode of vibration of the membrane corresponds to a specific eigenvalue pair (α_n, β_m), and the total solution is a superposition of all such modes.

20.11 Nodal Lines and Mode Shapes

For a given pair (m, n), the displacement function is:

$$u_{mn}(x, y, t) = [A_{mn}\cos(\omega_{mn}t) + B_{mn}\sin(\omega_{mn}t)] \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)$$

The **nodal lines** are those lines in the membrane where $u_{mn}(x, y, t) = 0$ for all t. These occur when:

- $\sin\left(\frac{n\pi x}{a}\right) = 0$, i.e., at $x = \frac{ka}{n}$, for $k = 0, 1, ..., n$
- $\sin\left(\frac{m\pi y}{b}\right) = 0$, i.e., at $y = \frac{lb}{m}$, for $l = 0, 1, ..., m$

The shape of the vibration pattern formed by these nodal lines is called the **mode shape**.

For example:

- The mode $(1, 1)$ has no interior nodal lines — only boundary constraints.
- The mode $(2, 1)$ will have a nodal line along $x = \frac{a}{2}$, dividing the membrane vertically.
- The mode $(1, 2)$ will have a nodal line along $y = \frac{b}{2}$, dividing it horizontally.

These shapes can be visualized using contour plots or physical simulations in civil engineering modeling software.

20.12 Forced Vibrations and Damping (Overview)

While the basic model assumes **free vibration** (i.e., no external force or energy loss), real-world membranes often experience:

- **Forced vibration**: where external periodic forces act on the system.
- **Damping**: due to internal friction or air resistance, reducing amplitude over time.

For a forced system:

$$\frac{\partial^2 u}{\partial t^2} + 2h\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + F(x, y, t)$$

where:

- h is the damping coefficient,
- $F(x, y, t)$ is the external force distribution.

Though more complex, such equations can also be approached using **Fourier series**, **Laplace transforms**, or **numerical methods** like finite difference or finite element analysis — all vital in advanced civil engineering simulations.

20.13 Computational Considerations

In practical engineering:

- **Only the first few terms** of the double Fourier series are used for an **approximate solution**.
- **Truncating** the series after a finite number of terms allows for numerical computation and modeling.
- **Software tools** (like MATLAB, ANSYS, Abaqus) use such series approximations or numerical PDE solvers to model membrane and plate behavior under dynamic loads.

Example:

To compute the displacement at a certain point, a civil engineer might use:

$$u(x_0, y_0, t_0) \approx \sum_{n=1}^{N} \sum_{m=1}^{M} [A_{mn} \cos(\omega_{mn} t_0) + B_{mn} \sin(\omega_{mn} t_0)] \sin\left(\frac{n\pi x_0}{a}\right) \sin\left(\frac{m\pi y_0}{b}\right)$$

for a selected number of terms N, M.

20.14 Practical Problems in Civil Engineering Using Double Fourier Series

Some real-life problems where this mathematical framework is used include:

- **Analysis of roof vibrations** during earthquakes.
- **Design of stadium canopies** and large fabric structures.
- **Response of suspended pedestrian bridges** to wind or human-induced vibrations.
- **Simulation of soil surface vibration patterns** for geotechnical investigations.

Each of these applications may involve rectangular (or near-rectangular) domains, fixed boundaries, and vibrating responses modeled through the Fourier method.
